

# THE LEHMER INEQUALITY AND THE MORDELL-WEIL THEOREM FOR DRINFELD MODULES

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ABSTRACT. In this paper we prove several Lehmer type inequalities for Drinfeld modules which will enable us to prove certain Mordell-Weil type structure theorems for Drinfeld modules.

*Keywords:* Mordell-Weil theorem, Drinfeld modules, Heights.

## 1. INTRODUCTION

The classical Lehmer conjecture (see [12], page 476) asserts that there is an absolute constant  $C > 0$  so that any algebraic number  $\alpha$  that is not a root of unity satisfies the following inequality for its logarithmic height

$$h(\alpha) \geq \frac{C}{[\mathbb{Q}(\alpha) : \mathbb{Q}]}.$$

A partial result towards this conjecture is obtained in [5]. The analog of Lehmer conjecture for elliptic curves and abelian varieties asks for a good lower bound for the canonical height of a non-torsion point of the abelian variety. Also this question has been much studied (see [1], [2], [11], [13], [17]).

In this paper we prove several inequalities for the height of non-torsion points of Drinfeld modules. These inequalities have the same flavor as the above mentioned Lehmer's conjecture. Using our inequalities we will be able to prove several Mordell-Weil type structure theorems for Drinfeld modules over certain infinitely generated fields. Next we will define the notion of Drinfeld modules.

In this paper we will use the following notation:  $p$  is a prime number and  $q$  is a power of  $p$ . We denote by  $\mathbb{F}_q$  the finite field with  $q$  elements. We let  $C$  be a nonsingular projective curve defined over  $\mathbb{F}_q$  and we fix a closed point  $\infty$  on  $C$ . Then we define  $A$  as the ring of functions on  $C$  that are regular everywhere except possibly at  $\infty$ .

We let  $K$  be a field extension of  $\mathbb{F}_q$ . We fix a morphism  $i : A \rightarrow K$ . We define the operator  $\tau$  as the power of the usual Frobenius with the property that for every  $x \in K^{\text{alg}}$ ,  $\tau(x) = x^q$ . Then we let  $K\{\tau\}$  be the ring of polynomials in  $\tau$  with coefficients from  $K$  (the addition is the usual one while the multiplication is the composition of functions).

We fix an algebraic closure of  $K$ , denoted  $K^{\text{alg}}$ . We denote by  $\mathbb{F}_p^{\text{alg}}$  the algebraic closure of  $\mathbb{F}_p$  inside  $K^{\text{alg}}$ . Also, for us, the symbol " $\subset$ " means inclusion, not necessarily strict inclusion.

A Drinfeld module is a morphism  $\phi : A \rightarrow K\{\tau\}$  for which the coefficient of  $\phi_a$  is  $i(a)$  for every  $a \in A$ , and there exists  $a \in A$  such that  $\phi_a \neq i(a)\tau^0$ . Following the definition from [10] we call  $\phi$  a Drinfeld module of generic characteristic if  $\ker(i) = \{0\}$  and we call  $\phi$  a Drinfeld module of finite characteristic if  $\ker(i) \neq \{0\}$ .

For each field  $L$  containing  $K$ ,  $\phi(L)$  denotes  $L$  with the  $A$ -action given by  $\phi$ . When  $K$  is a finitely generated extension of  $\mathbb{F}_q$ , it was proved in [14] (in the case that  $\text{trdeg}_{\mathbb{F}_p} K = 1$ ) and in [18] (for arbitrary finite, positive transcendence degree) that  $\phi(K)$  is the direct sum of a finite torsion submodule with a free submodule of rank  $\aleph_0$ . We will prove in Theorem 5.7 a similar structure result for certain infinitely generated extensions of  $\mathbb{F}_q$ .

The key result is Theorem 4.15, which will be proved in the fourth section of this paper. Mainly what will be needed for our result will be a better understanding of the heights associated to  $\phi$ , both local and global heights. These heights were first introduced in [3] and then contributions towards their understanding were done in [7], [14] and [18]. We mention that our results are part of our Ph.D. thesis [7]. Lemma 4.14 appears also in our paper [8], in which we prove a local Lehmer inequality for Drinfeld modules.

Theorem 4.15 will also give us the technical ingredient to obtain an uniform boundedness result for the torsion submodule of  $\phi(K)$ . This will be explained in Corollary 4.22.

## 2. HEIGHTS ASSOCIATED TO A DRINFELD MODULE

We continue with the notation from Section 1. So,  $K$  is a field extension of  $\mathbb{F}_q$  and  $\phi : A \rightarrow K\{\tau\}$  is a Drinfeld module. We define  $M_K$  as the set of all discrete valuations of  $K$ . We also normalize all the valuations  $v \in M_K$  such that the range of  $v$  is  $\mathbb{Z}$ .

**Definition 2.1.** We call a subset  $U \subset M_K$  equipped with a function  $d : U \rightarrow \mathbb{R}_{>0}$  a *good set of valuations* if the following properties are satisfied

- (i) for every nonzero  $x \in K$ , there are finitely many  $v \in U$  such that  $v(x) \neq 0$ .
- (ii) for every nonzero  $x \in K$ ,

$$\sum_{v \in U} d(v) \cdot v(x) = 0.$$

The positive real number  $d(v)$  will be called the *degree* of the valuation  $v$ . When we say that the positive real number  $d(v)$  is associated to the valuation  $v$ , we understand that the degree of  $v$  is  $d(v)$ .

When  $U$  is a good set of valuations, we will refer to property (ii) as the sum formula for  $U$ .

**Definition 2.2.** Let  $U$  be a good set of valuations on  $K$ . The set  $\{0\} \cup \{x \in K \mid v(x) = 0 \text{ for all } v \in U\}$  is the set of *constants* for  $U$ . We denote this set by  $C(U)$ .

**Lemma 2.3.** *Let  $U$  be a good set of valuations on  $K$ . If  $x \in K$  is integral at all places  $v \in U$ , then  $x \in C(U)$ .*

*Proof.* Let  $x \in K \setminus \{0\}$ . By the sum formula for  $U$ , if  $v(x) \geq 0$  for all  $v \in U$ , then actually  $v(x) = 0$  for all  $v \in U$  (a sum of non-negative numbers is 0 if and only if all the numbers are 0). □

**Lemma 2.4.** *Let  $U$  be a good set of valuations on a field  $K$ . The set  $C(U)$  is a subfield of  $K$ .*

*Proof.* By its definition,  $C(U)$  is closed under multiplication and division. Because of Lemma 2.3,  $C(U)$  is closed also under addition.  $\square$

**Definition 2.5.** Let  $v \in M_K$  of degree  $d(v)$ . We say that the valuation  $v$  is coherent (on  $K^{\text{alg}}$ ) if for every finite extension  $L$  of  $K$ ,

$$(1) \quad \sum_{\substack{w \in M_L \\ w|v}} e(w|v)f(w|v) = [L : K],$$

where  $e(w|v)$  is the ramification index and  $f(w|v)$  is the relative degree between the residue field of  $w$  and the residue field of  $v$ .

Condition (1) says that  $v$  is *defectless* in  $L$ . In this case, we also let the degree of any  $w \in M_L$ ,  $w|v$  be

$$(2) \quad d(w) = \frac{f(w|v)d(v)}{[L : K]}.$$

It is immediate to see that (2) is equivalent to the stronger condition that for every two finite extensions of  $K$ ,  $L_1 \subset L_2$  and for every  $v_2 \in M_{L_2}$  that lies over  $v_1 \in M_{L_1}$ , which in turn lies over  $v$ ,

$$(3) \quad d(v_2) = \frac{f(v_2|v_1)d(v_1)}{[L_2 : L_1]}.$$

We will use in our proofs the following result from [6] (see (18.1), page 136).

**Lemma 2.6.** *Let  $L_1 \subset L_2 \subset L_3$  be a tower of finite extensions. Let  $v \in M_{L_1}$  and denote by  $w_1, \dots, w_s$  all the places of  $L_2$  that lie over  $v$ . Then the following two statements are equivalent:*

1)  $v$  is defectless in  $L_3$ .

2)  $v$  is defectless in  $L_2$  and  $w_1, \dots, w_s$  are defectless in  $L_3$ .

Lemma 2.6 shows that condition (1) of Definition 2.5 is equivalent to the following statement: for every two finite extensions of  $K$ ,  $L_1 \subset L_2$  and for every  $v_1 \in M_{L_1}$ ,  $v_1|v$

$$(4) \quad \sum_{\substack{w \in M_{L_2} \\ w|v_1}} e(w|v_1) f(w|v_1) = [L_2 : L_1].$$

The following result is an immediate consequence of Definition 2.5 and Lemma 2.6.

**Lemma 2.7.** *If  $v \in M_K$  is a coherent valuation (on  $K^{\text{alg}}$ ), then for every finite extension  $L$  of  $K$  and for every  $w \in M_L$  and  $w|v$ ,  $w$  is a coherent valuation (on  $K^{\text{alg}} = L^{\text{alg}}$ ).*

**Definition 2.8.** We let  $U_K$  be a good set of valuations on  $K$ . We call  $U_K$  a *coherent good set of valuations* (on  $K^{\text{alg}}$ ) if the following two conditions are satisfied

(i) for every finite extension  $L$  of  $K$ , if  $U_L \subset M_L$  is the set of all valuations lying over valuations from  $U_K$ , then  $U_L$  is a good set of valuations.

(ii) for every  $v \in U_K$ , the valuation  $v$  is coherent (on  $K^{\text{alg}}$ ).

*Remark 2.9.* Using the argument from page 9 of [15], we conclude that condition (i) from Definition 2.8 is automatically satisfied if  $U_K$  is a good set of valuations and if condition (ii) of Definition 2.8 is satisfied.

An immediate corollary to Lemma 2.7 is the following result.

**Corollary 2.10.** *If  $U_K \subset M_K$  is a good set of valuations that is coherent (on  $K^{\text{alg}}$ ), then for every finite extension  $L$  of  $K$ , if  $U_L$  is the set of all valuations on  $L$  which lie over valuations from  $U_K$ , then  $U_L$  is a coherent good set of valuations.*

Fix now a field  $K$  of characteristic  $p$  and let  $\phi : A \rightarrow K\{\tau\}$  be a Drinfeld module. Let  $v \in M_K$  be a coherent valuation (on  $K^{\text{alg}}$ ). Let  $d(v)$  be the degree of  $v$  as in Definition 2.5. For such  $v$ , we construct the local height  $\widehat{h}_v$  with respect to the Drinfeld module  $\phi$ . Our construction follows [14]. For  $x \in K$ , we set  $\tilde{v}(x) = \min\{0, v(x)\}$ . For a non-constant element  $a \in A$ , we define

$$(5) \quad V_v(x) = \lim_{n \rightarrow \infty} \frac{\tilde{v}(\phi_{a^n}(x))}{\deg(\phi_{a^n})}.$$

This function is well-defined and satisfies the same properties as in Propositions 1-3 from [14]. Mainly, we will use the following facts:

- 1) if  $x$  and all the coefficients of  $\phi_a$  are integral at  $v$ , then  $V_v(x) = 0$ .
- 2) for all  $b \in A \setminus \{0\}$ ,  $V_v(\phi_b(x)) = \deg(\phi_b) \cdot V_v(x)$ . Moreover, we can use any non-constant  $a \in A$  for the definition of  $V_v(x)$  and we will always get the same function  $V_v$ .
- 3)  $V_v(x \pm y) \geq \min\{V_v(x), V_v(y)\}$ .
- 4) if  $x \in \phi_{\text{tor}}$ , then  $V_v(x) = 0$ .

We define then

$$(6) \quad \widehat{h}_v(x) = -d(v)V_v(x).$$

If  $L$  is a finite extension of  $K$  and  $w \in M_L$  lies over  $v$  then we define similarly the function  $V_w$  on  $L$  and just as above, we let  $\widehat{h}_w(x) = -d(w)V_w(x)$  for every  $x \in L$ .

If  $U = U_K \subset M_K$  is a coherent good set of valuations, then for each  $v \in U$ , we denote by  $\widehat{h}_{U,v}$  the local height associated to  $\phi$  with respect to  $v$  (the construction of  $\widehat{h}_{U,v}$  is identical with the one from above). Then we define the global height associated to  $\phi$  as

$$(7) \quad \widehat{h}_U(x) = \sum_{v \in U} \widehat{h}_{U,v}(x).$$

For each  $x$ , the above sum is finite due to fact 1) stated above (see also Proposition 6 of [14]).

For each finite extension  $L$  of  $K$ , we let  $U_L$  be the set of all valuations of  $L$  that lie over places from  $U_K$ . As stated in Corollary 2.10,  $U_L$  is also a coherent good set of valuations and so, we can define the local heights  $\widehat{h}_{U_L, w}$  with respect to  $w \in U_L$ , associated to  $\phi$  for all elements  $x \in L$ . Then we define the global height of  $x$  as

$$\widehat{h}_{U_L}(x) = \sum_{w \in U_L} \widehat{h}_{U_L, w}(x).$$

**Claim 2.11.** Let  $L_1 \subset L_2$  be finite extensions of  $K$ . Let  $v \in U_{L_1}$  and  $x \in L_1$ . Then

$$\sum_{\substack{w \in U_{L_2} \\ w|v}} \widehat{h}_{U_{L_2}, w}(x) = \widehat{h}_{U_{L_1}, v}(x).$$

*Proof.* We have

$$\sum_{\substack{w \in U_{L_2} \\ w|v}} \widehat{h}_{U_{L_2}, w}(x) = - \sum_{\substack{w \in U_{L_2} \\ w|v}} d(w) V_w(x).$$

Because  $d(w) = \frac{d(v)f(w|v)}{[L_2:L_1]}$  (see (3)) and  $V_w(x) = e(w|v)V_v(x)$  we get

$$\sum_{\substack{w \in U_{L_2} \\ w|v}} \widehat{h}_{U_{L_2}, w}(x) = \frac{-d(v)V_v(x)}{[L_2:L_1]} \sum_{\substack{w \in U_{L_2} \\ w|v}} e(w|v)f(w|v).$$

Because  $v$  is defectless and  $\widehat{h}_{U_{L_1}, v}(x) = -d(v)V_v(x)$ , we are done.  $\square$

Claim 2.11 shows that our definition of the global height is independent of the field  $L$  containing  $x$  and so, we can drop the index referring to the field  $L$  containing  $x$  when we work with the global height associated to a coherent good set of valuations.

The above construction for global heights depends on the selected good set of valuations  $U_K$  on  $K$ . If we work with global heights only for points  $x \in K$ , then  $U_K$  can be any good

set of valuations on  $K$ . If we are interested in global heights for all points  $x \in K^{\text{alg}}$ , then  $U_K$  has to be a good set of valuations on  $K$ , which is coherent (on  $K^{\text{alg}}$ ). Also, technically speaking, we do not need for local heights  $\widehat{h}_v$  the valuation  $v$  be coherent as long as we restrict ourselves to points  $x \in K$ . We will always specify first which is the good set of valuations that we consider when we will work with heights associated to a Drinfeld module.

### 3. EXAMPLES OF COHERENT GOOD SETS OF VALUATIONS

Let  $F$  be a field of characteristic  $p$  and let  $K = F(x_1, \dots, x_n)$  be the rational function field of transcendence degree  $n \geq 1$  over  $F$ . We let  $F^{\text{alg}}$  be the algebraic closure of  $F$  inside  $K^{\text{alg}}$ . We will construct a coherent good set of valuations on  $K$ .

First we construct a good set of valuations on  $K$  and then we will show that this set is also coherent. According to Remark 2.9, we only need to show that each of the valuations on  $K$  we construct is coherent.

Let  $R = F[x_1, \dots, x_n]$ . For each irreducible polynomial  $P \in R$  we let  $v_P$  be the valuation on  $K$  given by

$$v_P\left(\frac{Q_1}{Q_2}\right) = \text{ord}_P(Q_1) - \text{ord}_P(Q_2) \text{ for every nonzero } Q_1, Q_2 \in R,$$

where by  $\text{ord}_P(Q)$  we denote the order of the polynomial  $Q \in R$  at  $P$ .

Also, we construct the valuation  $v_\infty$  on  $K$  given by

$$v_\infty\left(\frac{Q_1}{Q_2}\right) = \deg(Q_2) - \deg(Q_1) \text{ for every nonzero } Q_1, Q_2 \in R,$$

where by  $\deg(Q)$  we denote the total degree of the polynomial  $Q \in R$ .

We let  $M_{K/F}$  be the set of all valuations  $v_P$  for irreducible polynomials  $P \in R$  plus the valuation  $v_\infty$ . We let the degree of  $v_P$  be  $d(v_P) = \deg(P)$  for every irreducible polynomial

$P \in R$  and we also let  $d(v_\infty) = 1$ . Then, for every nonzero  $x \in K$ , we have the sum formula

$$\sum_{v \in M_{K/F}} d(v) \cdot v(x) = 0.$$

So,  $M_{K/F}$  is a good set of valuations on  $K$  according to Definition 2.1. The field  $F$  is the field of constants with respect to  $M_{K/F}$ .

*Remark 3.1.* The valuations constructed above are exactly the valuations associated with the irreducible divisors of the projective space  $\mathbb{P}_F^n$ . The degrees of the valuations are the projective degrees of the corresponding irreducible divisors.

Let  $K'$  be a finite extension of  $K$  and let  $F'$  be the algebraic closure of  $F$  in  $K'$ . We let  $M_{K'/F'}$  be the set of all valuations on  $K'$  that extend the valuations from  $M_{K/F}$ . We normalize each valuation  $w$  from  $M_{K'/F'}$  so that the range of  $w$  is  $\mathbb{Z}$ . Also, we define

$$(8) \quad d(w) = \frac{f(w|v)d(v)}{[K' : K]}$$

for every  $w \in M_{K'/F'}$  and  $v \in M_{K/F}$  such that  $w|v$ . Note that strictly speaking,  $w$  is an extension of  $v$  as a place and *not* as a valuation function. However, we still call  $w$  an extension of  $v$ .

*Remark 3.2.* Continuing the observations made in Remark 3.1, the valuations defined on  $K'$  are the ones associated with irreducible divisors of the normalization of  $\mathbb{P}_F^n$  in  $K'$ . In general, the discrete valuations associated with the irreducible divisors of a variety which is regular in codimension 1 form a coherent good set of valuations.

In order to show that  $M_{K/F}$  is a coherent good set of valuations (on  $K^{\text{alg}}$ ), we need to check that condition (1) of Definition 2.5 is satisfied. This is proved in Chapter 1, Section

4 of [16] (Hypothesis (F) holds for algebras of finite type over fields and so, it holds for localizations of such algebras). For each  $v \in M_{K/F}$  we apply Propositions 10 and 11 of [16] to the local ring of  $v$  to show  $v$  is coherent.

Now, in general, let  $F$  be a field of characteristic  $p$  and let  $K$  be any finitely generated extension over  $F$ , of positive transcendence degree over  $F$ . If  $F$  is algebraically closed in  $K$ , we construct a coherent good set of valuations  $M_{K/F} \subset M_K$ , as follows. We pick a transcendence basis  $\{x_1, \dots, x_n\}$  for  $K/F$  and first construct as before the set of valuations on  $F(x_1, \dots, x_n)$ :

$$\{v_\infty\} \cup \{v_P \mid P \text{ irreducible polynomial in } F[x_1, \dots, x_n]\}.$$

Then, by Corollary 2.10, we have a unique way of extending coherently this set of valuations to a good set of valuations on  $K$ . The set  $M_{K/F}$  depends on our initial choice of the transcendence basis for  $K/F$ . Thus, in our notation  $M_{K/F}$ , we suppose that  $K/F$  comes equipped with a choice of a transcendence basis for  $K/F$ .

We also note that for every  $v \in M_{K/F}$ , if  $v_0 \in M_{F(x_1, \dots, x_n)/F}$  lies below  $v$ , then

$$(9) \quad d(v) = \frac{f(v|v_0)d(v_0)}{[K : F(x_1, \dots, x_n)]} \geq \frac{1}{[K : F(x_1, \dots, x_n)]}.$$

In general, if  $K'$  is a finite extension of  $K$  and  $v' \in M_{K'}$  lies above  $v \in M_K$ , then

$$(10) \quad d(v') = \frac{f(v'|v)d(v)}{[K' : K]} \geq \frac{d(v)}{[K' : K]}.$$

For each such good set of valuations  $M_{K/F}$  and for any Drinfeld module  $\phi : A \rightarrow K\{\tau\}$ , we construct as before the set of local heights and the global height associated to  $\phi$ . We denote the local heights by  $\widehat{h}_{M_{K/F}, v}$  and the global height by  $\widehat{h}_{M_{K/F}}$ . If  $F$  is a finite field, our construction coincides with the one from [18]. Thus, if  $F$  is a finite field, we will drop the

subscript  $M_{K/F}$  from the notation of the local heights and of the global height. Also, when  $F$  is a finite field and  $\text{trdeg}_F K = 1$ , our construction also coincides with the one from [14].

#### 4. LEHMER INEQUALITY FOR DRINFELD MODULES

The paper [4] formulated a conjecture whose general form is Conjecture 4.1, which we refer to as the Lehmer inequality for Drinfeld modules.

**Conjecture 4.1.** Let  $K$  be a finitely generated field. For any Drinfeld module  $\phi : A \rightarrow K\{\tau\}$  there exists a constant  $C > 0$  depending only on  $\phi$  such that any non-torsion point  $x \in K^{\text{alg}}$  satisfies  $\widehat{h}(x) \geq \frac{C}{[K(x):K]}$ .

We fix a non-constant element  $t \in A$  and we let

$$\phi_t = \sum_{i=0}^r a_i \tau^i.$$

The statement of Conjecture 4.1 is not affected if we replace  $K$  by a finite extension  $K'$  since if we find a constant  $C'$  that works for  $K'$  in Conjecture 4.1, then  $C = \frac{C'}{[K':K]}$  will work for  $K$ .

If we conjugate  $\phi$  by  $\gamma \in K^{\text{alg}} \setminus \{0\}$  (i.e.  $a \rightarrow \gamma^{-1}\phi_a\gamma$  for every  $a \in A$ ), we obtain a new Drinfeld module, which we denote by  $\phi^{(\gamma)}$  and these two Drinfeld modules are isomorphic over  $K(\gamma)$ . As a particular case of Proposition 2 of [14] we get that  $\widehat{h}_\phi(x) = \widehat{h}_{\phi^{(\gamma)}}(\gamma^{-1}x)$ . Then, if Conjecture 4.1 is proved for  $\phi^{(\gamma)}$ , it will also hold for  $\phi$ . So, having these in mind, we replace  $\phi$  by one of its conjugates that has the property that  $\phi_t^{(\gamma)}$  is monic, i.e. with the above notations,  $\gamma$  satisfies the equation  $\gamma^{q^r-1}a_r = 1$ . Because  $[K(\gamma) : K] \leq q^r - 1$ , working over  $K(\gamma)$  instead of  $K$ , we may introduce a factor of  $\frac{1}{q^r-1}$  at the worst in the constant  $C$  from Conjecture 4.1, as explained in the previous paragraph. Also, the module structure

theorems that we will be proving in the next section will not be affected by replacing  $\phi$  with an isomorphic Drinfeld module or by replacing  $K$  with a finite extension.

So, for simplifying the notation we suppose from now on in this section that  $\phi_t$  is monic.

In this section we will prove Theorem 4.5, which is a special case of the Lehmer inequality for Drinfeld modules. We will actually prove a more general result (Theorem 4.15) from which we will be able to infer Theorems 4.4 and 4.5.

For each finite extension  $L$  of  $K$ , we let  $S_L$  be the set of places  $v \in M_L$  for which there exists a coefficient  $a_i$  of  $\phi_t$  such that  $v(a_i) < 0$ . We will prove that the set  $S_L$  is the set of all valuations on  $L$  of *bad reduction* for  $\phi$ . We define next the notion of *good reduction* for a Drinfeld module.

**Definition 4.2.** Let  $\phi : A \rightarrow K\{\tau\}$  be a Drinfeld module. Let  $L$  be a finite extension of  $K$ . We call  $v \in M_L$  a place of *good reduction* for  $\phi$  if for all  $a \in A \setminus \{0\}$ , the coefficients of  $\phi_a$  are integral at  $v$  and the leading coefficient of  $\phi_a$  is a unit in the valuation ring at  $v$ . If  $v \in M_L$  is not a place of good reduction, we call it a place of *bad reduction*.

**Lemma 4.3.** *The set  $S_L$  is the set of all places from  $M_L$  at which  $\phi$  has bad reduction.*

*Proof.* By the construction of the set  $S_L$ , the places from  $S_L$  are of bad reduction for  $\phi$ . We will prove that these are all the bad places for  $\phi$ .

Let  $a \in A$ . The equation  $\phi_a \phi_t = \phi_t \phi_a$  will show that all the places where not all of the coefficients of  $\phi_a$  are integral, are from  $S_L$ . Suppose this is not the case and take a place  $v \notin S_L$  at which some coefficient of  $\phi_a$  is not integral. Let  $\phi_a = \sum_{i=0}^{r'} a'_i \tau^i$  and assume that  $i$  is the largest index for a coefficient  $a'_i$  that is not integral at  $v$ .

We equate the coefficient of  $\tau^{i+r}$  in  $\phi_a\phi_t$  and  $\phi_t\phi_a$ , respectively. The former is

$$(11) \quad a'_i + \sum_{j>i} a'_j a_{r+i-j}^{q^j}$$

while the latter is

$$(12) \quad a_i'^{q^r} + \sum_{j>i} a_{r+i-j} a_j'^{q^{r+i-j}}.$$

Thus the valuation at  $v$  of (11) is  $v(a'_i)$ , because all the  $a'_j$  (for  $j > i$ ) and  $a_{r+i-j}$  are integral at  $v$ , while  $v(a'_i) < 0$ . Similarly, the valuation of (12) is  $v(a_i'^{q^r}) = q^r v(a'_i) < v(a'_i)$  ( $r \geq 1$  because  $t$  is non-constant). This fact gives a contradiction to  $\phi_a\phi_t = \phi_t\phi_a$ . So, the coefficients of  $\phi_a$  for all  $a \in A$ , are integral at all places of  $M_L \setminus S_L$ .

Now, using the same equation  $\phi_a\phi_t = \phi_t\phi_a$  and equating the leading coefficients in both polynomials we obtain

$$a'_{r'} = a_{r'}'^{q^{r'}}.$$

So,  $a'_{r'} \in \mathbb{F}_p^{\text{alg}}$ . Thus, all the leading coefficients for polynomials  $\phi_a$  are constants with respect to the valuations of  $L$ . So, if  $v \in M_L \setminus S_L$ , then all the coefficients of  $\phi_a$  are integral at  $v$  and the leading coefficient of  $\phi_a$  is a unit in the valuation ring at  $v$  for every  $a \in A \setminus \{0\}$ . Thus,  $v \notin S_L$  is a place of good reduction for  $\phi$ .  $\square$

**Theorem 4.4.** *Let  $K$  be a finitely generated field of characteristic  $p$ . Let  $\phi : A \rightarrow K\{\tau\}$  be a Drinfeld module and assume that there exists a non-constant  $t \in A$  such that  $\phi_t$  is monic. Let  $F$  be the algebraic closure of  $\mathbb{F}_p$  in  $K$ . We let  $M_{K/F}$  be the coherent good set of valuations on  $K$ , constructed as in Section 3. Let  $\widehat{h}$  and  $\widehat{h}_v$  be the global and local heights associated to  $\phi$ , constructed with respect to the coherent good set of valuations  $M_{K/F}$ . Let  $x \in K^{\text{alg}}$  and let  $F_x$  be the algebraic closure of  $\mathbb{F}_p$  in  $K(x)$ . We construct the good set of valuations  $M_{K(x)/F_x}$*

which lie above the valuations from  $M_{K/F}$ . Let  $S_x$  be the set of places  $v \in M_{K(x)/F_x}$  such that  $\phi$  has bad reduction at  $v$ .

If  $x$  is not a torsion point for  $\phi$ , then there exists  $v \in M_{K(x)/F_x}$  such that

$$\widehat{h}_v(x) > q^{-r(2+(r^2+r)|S_x|)} d(v)$$

where  $d(v)$  is as always the degree of the valuation  $v$ .

Let  $\{x_1, \dots, x_n\}$  be the transcendence basis for  $K/F$  associated to the construction of  $M_{K/F}$ . Let  $v_0 \in M_{K/F}$  be the place lying below the place  $v$  from the conclusion of Theorem 4.4. Then  $d(v) = \frac{d(v_0)f(v|v_0)}{[K(x):K]}$ . Because  $f(v|v_0) \geq 1$ ,  $d(v_0) \geq \frac{1}{[K:F(x_1, \dots, x_n)]}$  (see (9)) and  $\widehat{h}(x) \geq \widehat{h}_v(x)$ , Theorem 4.4 has the following corollary.

**Theorem 4.5.** *With the notation from Theorem 4.4, if  $x \notin \phi_{\text{tor}}$ , then*

$$\widehat{h}(x) > \frac{q^{-r(2+(r^2+r)|S_x|)}}{[K(x) : F(x_1, \dots, x_n)]}.$$

*Remark 4.6.* Theorem 4.5 is a weaker form of Conjecture 4.1 because our constant  $C$  for which  $\widehat{h}(x) \geq \frac{C}{[K(x):K]}$  for  $x \notin \phi_{\text{tor}}$ , is not completely independent of  $K(x)$ . For us,

$$C = \frac{q^{-r(2+(r^2+r)|S_x|)}}{[K : F(x_1, \dots, x_n)]}$$

and  $S_x$  depends on  $K(x)$ .

Before proving Theorem 4.4, we need to prove several preliminary lemmas regarding the local height for an arbitrary point of the Drinfeld module  $\phi$ .

Fix now a finite extension  $L$  of  $K$  and let  $U$  be a good set of valuations on  $L$ . Let  $S = S_L \cap U$ .

For each  $v \in U$  we define

$$(13) \quad M_v = \min_{i \in \{0, \dots, r-1\}} \frac{v(a_i)}{q^r - q^i}$$

where by convention, as always:  $v(0) = +\infty$ . We observe that  $M_v < 0$  if and only if  $v \in S$ .

Let  $v \in S$ . We define  $P_v$  as the subset of the negatives of the slopes of the Newton polygon associated to  $\phi_t$ , consisting of those  $\alpha$  for which there exist  $i \neq j$  in  $\{0, \dots, r\}$  such that

$$(14) \quad \alpha = \frac{v(a_i) - v(a_j)}{q^j - q^i} \leq 0,$$

and  $v(a_i) + q^i \alpha = v(a_j) + q^j \alpha = \min_{0 \leq l \leq r} (v(a_l) + q^l \alpha)$ . If  $\phi$  is the Carlitz module in characteristic 2, i.e.  $\phi = \psi_2$ , where  $\psi_2 : \mathbb{F}_2[t] \rightarrow K\{\tau\}$  is defined by  $\psi_2(x) = tx + x^2$  for every  $x$ , then we want the set  $P_v$  to contain  $\{0\}$ , even if 0 is not in the set from (14).

Let

$$N_\phi = \begin{cases} 1 + r = 2, & \text{if } \phi = \psi_2 \\ r, & \text{otherwise.} \end{cases}$$

Clearly, for every  $\phi$  and  $v \in S$ ,  $|P_v| \leq N_\phi$ . We define next the concept of angular component for a nonzero  $x \in L$ . For this we first fix a uniformizer  $\pi_v \in L$  for each valuation  $v \in S$ .

**Definition 4.7.** Assume  $v \in S$ . For every nonzero  $y \in L$  we define the angular component of  $y$  at  $v$ , denoted by  $\text{ac}_{\pi_v}(y)$ , to be the residue at  $v$  of  $y\pi_v^{-v(y)}$ . (Note that the angular component is never 0.)

We can define in a similar manner as above the notion of angular component at each  $v \in M_L$  but we will work with angular components at the places from  $S$  only.

The main property of the angular component is that for every  $y, z \in L \setminus \{0\}$ ,  $v(y - z) > v(y) = v(z)$  if and only if  $(v(y), \text{ac}_{\pi_v}(y)) = (v(z), \text{ac}_{\pi_v}(z))$ .

For each  $\alpha \in P_v$  we let  $l \geq 1$  and let  $i_0 < i_1 < \dots < i_l$  be all the indices  $i$  for which

$$(15) \quad v(a_i \alpha^{q^i}) = \min_j v(a_j \alpha^{q^j}).$$

We define  $R_v(\alpha)$  as the set containing all the nonzero solutions of the equation

$$(16) \quad \sum_{j=0}^l \text{ac}_{\pi_v}(a_{i_j}) X^{q^{i_j}} = 0,$$

where the indices  $i_j$  are the ones associated to  $\alpha$  as in (15). For  $\alpha = 0$ , we want the set  $R_v(\alpha)$  to contain also  $\{1\}$  in addition to the numbers from (16). If  $\alpha = 0$ ,  $l$  might be 0 and so, equation (16) might have no nonzero solutions. In that case,  $R_v(0) = \{1\}$ . Clearly, for every  $\alpha \in P_v$ ,  $|R_v(\alpha)| \leq q^r$ .

**Lemma 4.8.** *Assume  $v \in S$  and let  $x \in L$ . If  $v(x) \leq 0$  and  $v(\phi_t(x)) > \min_{i \in \{0, \dots, r\}} v(a_i x^{q^i})$  then  $(v(x), \text{ac}_{\pi_v}(x)) \in P_v \times R_v(v(x))$ .*

*Proof.* If  $v(\phi_t(x)) > \min_{i \in \{0, \dots, r\}} v(a_i x^{q^i})$  it means that there exists  $l \geq 1$  and

$$i_0 < \dots < i_l$$

such that

$$(17) \quad v(a_{i_0} x^{q^{i_0}}) = \dots = v(a_{i_l} x^{q^{i_l}}) = \min_{i \in \{0, \dots, r\}} v(a_i x^{q^i})$$

and also

$$(18) \quad \sum_{j=0}^l \text{ac}_{\pi_v}(a_{i_j}) \text{ac}_{\pi_v}(x)^{q^{i_j}} = 0.$$

Equations (17) and (18) yield  $v(x) \in P_v$  and  $\text{ac}_{\pi_v}(x) \in R_v(v(x))$  respectively, according to (14) and (16).

Note that we needed our assumption that  $v(x) \leq 0$  only because  $P_v$  consists only of the negatives of the non-negative slopes of the Newton polygon associated to  $\phi_t$  (and not the negatives of all the slopes). The above proof shows that as long as the valuation of  $x$  and the angular component of  $x$  do not belong to certain prescribed sets,  $v(\phi_t(x)) = \min_i v(a_i x^{q^i})$ .  $\square$

**Lemma 4.9.** *Let  $v \in M_L$  and let  $x \in L$ . If  $v(x) < \min\{0, M_v\}$ , then  $\widehat{h}_v(x) = -d(v) \cdot v(x)$ .*

*Proof.* For every  $i \in \{0, \dots, r-1\}$ ,  $v(a_i x^{q^i}) = v(a_i) + q^i v(x) > q^r v(x)$  because  $v(x) < M_v = \min_{i \in \{0, \dots, r-1\}} \frac{v(a_i)}{q^r - q^i}$ . This shows that  $v(\phi_t(x)) = q^r v(x) < v(x) < \min\{0, M_v\}$ . By induction,  $v(\phi_{t^n}(x)) = q^{rn} v(x)$  for all  $n \geq 1$ . So,  $V_v(x) = v(x)$  and

$$\widehat{h}_v(x) = -d(v) \cdot v(x).$$

$\square$

An immediate corollary to (4.9) is the following result.

**Lemma 4.10.** *Assume  $v \notin S$  and let  $x \in L$ . If  $v(x) < 0$  then  $\widehat{h}_v(x) = -d(v) \cdot v(x)$ , while if  $v(x) \geq 0$  then  $\widehat{h}_v(x) = 0$ .*

*Proof.* First, it is clear that if  $v(x) \geq 0$  then for all  $n \geq 1$ ,  $v(\phi_{t^n}(x)) \geq 0$  because all the coefficients of  $\phi_t$  and thus of  $\phi_{t^n}$  have non-negative valuation at  $v$ . Thus  $V_v(x) = 0$  and so,

$$\widehat{h}_v(x) = 0.$$

Now, if  $v(x) < 0$ , then  $v(x) < M_v$  because  $M_v \geq 0$  ( $v \notin S$ ). So, applying the result of (4.9) we conclude the proof of this lemma.  $\square$

We will get a better insight into the local heights behaviour with the following lemma.

**Lemma 4.11.** *Let  $x \in L$ . Assume  $v \in S$  and  $v(x) \leq 0$ . If  $(v(x), \text{ac}_{\pi_v}(x)) \notin P_v \times R_v(v(x))$  then  $v(\phi_t(x)) < M_v$ , unless  $q = 2$ ,  $r = 1$  and  $v(x) = 0$ .*

*Proof.* Lemma 4.8 implies that there exists  $i_0 \in \{0, \dots, r\}$  such that for all  $i \in \{0, \dots, r\}$  we have  $v(a_i x^{q^i}) \geq v(a_{i_0} x^{q^{i_0}}) = v(\phi_t(x))$ .

Suppose (4.11) is not true and so, there exists  $j_0 < r$  such that

$$\frac{v(a_{j_0})}{q^r - q^{j_0}} \leq v(\phi_t(x)) = v(a_{i_0}) + q^{i_0} v(x).$$

This means that

$$(19) \quad v(a_{j_0}) \leq (q^r - q^{j_0})v(a_{i_0}) + (q^{r+i_0} - q^{i_0+j_0})v(x).$$

On the other hand, by our assumption about  $i_0$ , we know that  $v(a_{j_0} x^{q^{j_0}}) \geq v(a_{i_0} x^{q^{i_0}})$  which means that

$$(20) \quad v(a_{j_0}) \geq v(a_{i_0}) + (q^{i_0} - q^{j_0})v(x).$$

Putting together inequalities (19) and (20), we get

$$v(a_{i_0}) + (q^{i_0} - q^{j_0})v(x) \leq (q^r - q^{j_0})v(a_{i_0}) + (q^{r+i_0} - q^{i_0+j_0})v(x).$$

Thus

$$(21) \quad v(x)(q^{r+i_0} - q^{i_0+j_0} - q^{i_0} + q^{j_0}) \geq -v(a_{i_0})(q^r - q^{j_0} - 1).$$

But  $q^{r+i_0} - q^{i_0+j_0} - q^{i_0} + q^{j_0} = q^{r+i_0}(1 - q^{j_0-r} - q^{-r} + q^{j_0-r-i_0})$  and because  $j_0 < r$  and  $q^{j_0-r-i_0} > 0$ , we obtain

$$(22) \quad 1 - q^{j_0-r} - q^{-r} + q^{j_0-r-i_0} > 1 - q^{-1} - q^{-r} \geq 1 - 2q^{-1} \geq 0.$$

Also,  $q^r - q^{j_0} - 1 \geq q^r - q^{r-1} - 1 = q^{r-1}(q - 1) - 1 \geq 0$  with equality if and only if  $q = 2$ ,  $r = 1$  and  $j_0 = 0$ . We will analyze this case separately. So, as long as we are not in this special case, we do have

$$(23) \quad q^r - q^{j_0} - 1 > 0.$$

Now we have two possibilities (remember that  $v(x) \leq 0$ ):

(i)  $v(x) < 0$

In this case, (21), (22) and (23) tell us that  $-v(a_{i_0}) < 0$ . Thus,  $v(a_{i_0}) > 0$ . But we know from our hypothesis on  $i_0$  that  $v(a_{i_0}x^{q^{i_0}}) \leq v(x^{q^r})$  which is in contradiction with the combination of the following facts:  $v(x) < 0$ ,  $i_0 \leq r$  and  $v(a_{i_0}) > 0$ .

(ii)  $v(x) = 0$

Then another use of (21), (22) and (23) gives us  $-v(a_{i_0}) \leq 0$ ; thus  $v(a_{i_0}) \geq 0$ . This would mean that  $v(a_{i_0}x^{q^{i_0}}) \geq 0$  and this contradicts our choice for  $i_0$  because we know from the fact that  $v \in S$ , that there exists  $i \in \{0, \dots, r\}$  such that  $v(a_i) < 0$ . So, then we would have

$$v(a_i x^{q^i}) = v(a_i) < 0 \leq v(a_{i_0} x^{q^{i_0}}).$$

Thus, in either case (i) or (ii) we get a contradiction that proves the lemma except in the special case that we excluded above:  $q = 2$ ,  $r = 1$  and  $j_0 = 0$ . If we have  $q = 2$  and  $r = 1$  then

$$\phi_t(x) = a_0 x + x^2.$$

By the definition of  $S$  and because  $v \in S$ ,  $v(a_0) < 0$ . Also,  $M_v = v(a_0)$ .

If  $v(x) < 0$ , then either  $v(x) < M_v = v(a_0)$ , in which case again  $v(\phi_t(x)) < M_v$  (as shown in the proof of Lemma 4.9), or  $v(x) \geq M_v$ . In the latter case,

$$v(\phi_t(x)) = v(a_0x) = v(a_0) + v(x) < v(a_0) = M_v.$$

So, we see that indeed, only  $v(x) = 0$ ,  $q = 2$  and  $r = 1$  can make  $v(\phi_t(x)) \geq M_v$  in the hypothesis of (4.11).  $\square$

**Lemma 4.12.** *Assume  $v \in S$  and let  $x \in L$ . Excluding the case  $q = 2$ ,  $r = 1$  and  $v(x) = 0$ , we have that if  $v(x) \leq 0$  then either  $\widehat{h}_v(x) > \frac{-d(v)M_v}{q^r}$  or  $(v(x), ac_{\pi_v}(x)) \in P_v \times R_v(v(x))$ .*

*Proof.* If  $v(x) \leq 0$  then

$$\text{either: (i) } v(\phi_t(x)) < M_v ,$$

in which case by (4.9) we have that  $\widehat{h}_v(\phi_t(x)) = -d(v) \cdot v(\phi_t(x))$ . So, case (i) yields

$$(24) \quad \widehat{h}_v(x) = -d(v) \cdot \frac{v(\phi_t(x))}{\deg \phi_t} > -d(v) \cdot \frac{M_v}{q^r}$$

$$\text{or: (ii) } v(\phi_t(x)) \geq M_v ,$$

in which case, Lemma 4.11 yields

$$(25) \quad v(\phi_t(x)) > v(a_{i_0}x^{q^{i_0}}) = \min_{i \in \{0, \dots, r\}} v(a_i x^{q^i}).$$

Using (25) and Lemma 4.8 we conclude that case (ii) yields  $(v(x), ac_{\pi_v}(x)) \in P_v \times R_v(v(x))$ .  $\square$

Now we analyze the excluded case from Lemma 4.12.

**Lemma 4.13.** *Assume  $v \in S$  and let  $x \in L$ . If  $v(x) \leq 0$ , then either*

$$(v(x), \text{ac}_{\pi_v}(x)) \in P_v \times R_v(v(x))$$

$$\text{or } \widehat{\text{h}}_v(x) \geq \frac{-d(v)M_v}{q^r}.$$

*Proof.* Using the result of (4.12) we have left to analyze the case:  $q = 2$ ,  $r = 1$  and  $v(x) = 0$ .

As shown in the proof of (4.11), in this case  $\phi_t(x) = a_0x + x^2$  and

$$v(\phi_t(x)) = v(a_0) = M_v < 0.$$

Then, either  $v(\phi_{t^2}(x)) = v(\phi_t(x)^2) = 2M_v < M_v$  or  $v(\phi_{t^2}(x)) > v(a_0\phi_t(x)) = v(\phi_t(x)^2)$ . If the former case holds, then by (4.9),

$$\widehat{\text{h}}_v(\phi_{t^2}(x)) = -d(v) \cdot 2M_v$$

and so,

$$\widehat{\text{h}}_v(x) = \frac{-d(v) \cdot 2M_v}{4}.$$

If the latter case holds, i.e.  $v(\phi_t(\phi_t(x))) > v(a_0\phi_t(x)) = v(\phi_t(x)^2)$ , then  $\text{ac}_{\pi_v}(\phi_t(x))$  satisfies the equation

$$\text{ac}_{\pi_v}(a_0)X + X^2 = 0.$$

Because the angular component is never 0, it must be that  $\text{ac}_{\pi_v}(\phi_t(x)) = \text{ac}_{\pi_v}(a_0)$  (remember that we are working now in characteristic 2). But, because  $v(a_0x) < v(x^2)$  we can relate the angular component of  $x$  and the angular component of  $\phi_t(x)$  and so,

$$\text{ac}_{\pi_v}(a_0) = \text{ac}_{\pi_v}(\phi_t(x)) = \text{ac}_{\pi_v}(a_0x) = \text{ac}_{\pi_v}(a_0) \text{ac}_{\pi_v}(x).$$

This means  $\text{ac}_{\pi_v}(x) = 1$  and so, the excluded case amounts to a dichotomy similar to the one from (4.12): either  $(v(x), \text{ac}_{\pi_v}(x)) = (0, 1)$  or  $\widehat{h}_v(x) = \frac{-d(v)M_v}{2}$ . The definitions of  $P_v$  and  $R_v(\alpha)$  from (14) and (16) respectively, yield that  $(0, 1) \in P_v \times R_v(0)$ .  $\square$

Finally, we note that in (4.13) we have

$$-\frac{d(v)M_v}{q^r} = -\frac{d(v)e(v|v_0)M_{v_0}}{q^r}.$$

We have obtained the following dichotomy.

**Lemma 4.14.** *Assume  $v \in S$ . If  $v(x) \leq 0$  then either  $(v(x), \text{ac}_{\pi_v}(x)) \in P_v \times R_v(v(x))$  or  $\widehat{h}_{U,v}(x) \geq \frac{-M_v d(v)}{q^r}$ . Moreover, by its definition  $M_v < -\frac{1}{q^r}$  and so, if the above latter case holds for  $x$ , then  $\widehat{h}_{U,v}(x) > \frac{d(v)}{q^{2r}}$ .*

We will deduce Theorem 4.4 from the following more general result.

**Theorem 4.15.** *Let  $K$  be a field extension of  $\mathbb{F}_q$  and let  $\phi : A \rightarrow K\{\tau\}$  be a Drinfeld module. Let  $L$  be a finite field extension of  $K$ . Let  $t$  be a non-constant element of  $A$  and assume that  $\phi_t = \sum_{i=0}^r a_i \tau^i$  is monic. Let  $U$  be a good set of valuations on  $L$  and let  $C(U)$  be, as always, the field of constants with respect to  $U$ . Let  $S$  be the finite set of valuations  $v \in U$  such that  $\phi$  has bad reduction at  $v$ . Let  $x \in L$ .*

- a) If  $S$  is empty, then either  $x \in C(U)$  or there exists  $v \in U$  such that  $\widehat{h}_{U,v}(x) \geq d(v)$ .*
- b) If  $S$  is not empty, then either  $x \in \phi_{\text{tor}}$ , or there exists  $v \in U$  such that  $\widehat{h}_{U,v}(x) > q^{-2r-r^2 N_\phi |S|} d(v) \geq q^{-r(2+(r^2+r)|S|)} d(v)$ . Moreover, if  $S$  is not empty and  $x \in \phi_{\text{tor}}$ , then there exists a polynomial  $b(t) \in \mathbb{F}_q[t]$  of degree at most  $r N_\phi |S|$  such that  $\phi_{b(t)}(x) = 0$ .*

*Proof.* a) Assume  $S$  is empty.

Then either  $v(x) \geq 0$  for all  $v \in U$  or there exists  $v \in U$  such that  $v(x) < 0$ . If the latter statement is true, then Lemma 4.10 shows that for any valuation  $v \in U$  for which  $v(x) < 0$ , we have

$$\widehat{h}_{U,v} \geq d(v),$$

because  $v \notin S$  ( $S$  is empty).

Now, if the former statement is true, i.e.  $x$  is integral at all places from  $U$ , then by Lemma 2.3,  $x \in C(U)$ .

b) Assume  $S$  is not empty.

Let  $v \in S$ . We will use several times the following result.

**Lemma 4.16.** *Let  $L$  be a field extension of  $\mathbb{F}_q$  and let  $v$  be a discrete valuation on  $L$ . Let  $I$  be a finite set of integers. Let  $N$  be an integer greater or equal than all the elements of  $I$ . For each  $\alpha \in I$ , let  $R(\alpha)$  be a nonempty finite set of nonzero elements of the residue field at  $v$ . Let  $W$  be an  $\mathbb{F}_q$ -vector subspace of  $L$  with the property that for all  $w \in W$ ,  $(v(w), \text{ac}_{\pi_v}(w)) \in I \times R(v(w))$  whenever  $v(w) \leq N$ .*

*Let  $f$  be the smallest positive integer greater or equal than  $\max_{\alpha \in I} \log_q |R(\alpha)|$ . Then the  $\mathbb{F}_q$ -codimension of  $\{w \in W \mid v(w) > N\}$  is bounded above by  $|I|f$ .*

*Proof of Lemma 4.16.* Let  $i = |I|$ . Let  $\alpha_0 < \dots < \alpha_{i-1}$  be the elements of  $I$ , and let  $\alpha_i = N + 1$ . For  $0 \leq j \leq i$ , define  $W_j = \{w \in W \mid v(w) \geq \alpha_j\}$ . For  $0 \leq j < i$ , the hypothesis gives an injection

$$W_j/W_{j+1} \rightarrow R(\alpha_j) \cup \{0\}$$

taking  $w$  to the residue of  $w/\pi_v^{\alpha_j}$ . Thus

$$q^{\dim_{\mathbb{F}_q} W_j/W_{j+1}} \leq q^f + 1 < q^{f+1},$$

so  $\dim_{\mathbb{F}_q} W_j/W_{j+1} \leq f$  (note that we used the fact that  $f > 0$  in order to have the inequality  $q^f + 1 < q^{f+1}$ ). Summing over  $j$  gives  $\dim_{\mathbb{F}_q} W_0/W_i \leq if$ , as desired.  $\square$

We apply Lemma 4.16 with  $N = 0$ ,  $I = P_v$  and  $R(\alpha) = R_v(\alpha)$  for every  $\alpha \in P_v$ . Because  $|P_v| \leq N_\phi$  and  $|R_v(\alpha)| \leq q^r$  for every  $\alpha \in P_v$ , we obtain the following result.

**Fact 4.17.** *Let  $v \in S$ . Let  $W$  be an  $\mathbb{F}_q$ -subspace of  $L$  with the property that for all  $w \in W$ ,  $(v(w), \text{ac}_{\pi_v}(w)) \in P_v \times R_v(v(w))$  whenever  $v(w) \leq 0$ .*

*Then the  $\mathbb{F}_q$ -codimension of  $\{w \in W \mid v(w) > 0\}$  in  $W$  is bounded above by  $rN_\phi$ .*

We apply Fact 4.17 for each  $v \in S$  and we deduce the following two results.

**Fact 4.18.** *Let  $W$  be an  $\mathbb{F}_q$ -subspace of  $L$  such that  $(v(w), \text{ac}_{\pi_v}(w)) \in P_v \times R_v(v(w))$  whenever  $v \in S$ ,  $w \in W$  and  $v(w) \leq 0$ . Then the  $\mathbb{F}_q$ -codimension of*

$$\{w \in W \mid v(w) > 0 \text{ for all } v \in S\}$$

*in  $W$  is bounded above by  $rN_\phi|S|$ .*

**Fact 4.19.** *Let notation be as in Fact 4.18. If moreover,  $\dim_{\mathbb{F}_q} W > rN_\phi|S|$ , then there exists a nonzero  $w \in W$  such that  $v(w) > 0$  for all  $v \in S$ .*

Using the above facts we prove the following claim which is the key for obtaining the result of Theorem 4.15.

**Claim 4.20.** Assume  $|S| \geq 1$ . If  $W$  is an  $\mathbb{F}_q$ -subspace of  $L$  and  $\dim_{\mathbb{F}_q} W > rN_\phi|S|$ , then there exists  $w \in W$  and there exists  $v \in U$  such that  $\widehat{\text{h}}_{U,v}(w) > \frac{d(v)}{q^{2r}}$ .

*Proof of Claim 4.20.* If there exists  $v \in U \setminus S$  and  $w \in W$  such that  $v(w) < 0$ , then by Lemma 4.10,

$$\widehat{h}_v(w) \geq d(v) > \frac{d(v)}{q^{2r}}.$$

Thus, suppose from now on in the proof of Claim 4.20, that for every  $v \in U \setminus S$  and every  $w \in W$ ,  $v(w) \geq 0$ .

Because  $\dim_{\mathbb{F}_q} W > rN_\phi|S|$ , Fact 4.19 guarantees the existence of a nonzero  $w \in W$  for which *either*  $v(w) > 0$  for all  $v \in S$ , *or* there exists  $v \in S$  such that

$$(26) \quad v(w) \leq 0 \text{ but } (v(w), \text{ac}_{\pi_v}(w)) \notin P_v \times R_v(v(w)).$$

The former case is impossible because we already supposed that  $v(w) \geq 0$  for all  $v \in U \setminus S$ . Because  $|S| \geq 1$  there is no nonzero  $w$  that has non-negative valuation at all the places in  $U$  and positive valuation at at least one place in  $U$ . Its existence would contradict the sum formula for the valuations in  $U$ .

Thus, the latter case holds, i.e. there exists  $v \in S$  satisfying (26). But then, Lemma 4.14 gives  $\widehat{h}_{U,v}(w) > \frac{d(v)}{q^{2r}}$ . □

Using Claim 4.20 we can finish the proof of part b) of Theorem 4.15.

Consider  $W = \text{Span}_{\mathbb{F}_q} (\{x, \phi_t(x), \dots, \phi_{t^{rN_\phi|S|}}(x)\})$ . If there exists no polynomial  $b(t)$  as in the statement of part b) of Theorem 4.15, then  $\dim_{\mathbb{F}_q} W = 1 + rN_\phi|S|$ . Applying Claim 4.20 to  $W$ , we find some  $w \in W$  and some  $v \in U$  such that

$$(27) \quad \widehat{h}_{U,v}(w) > \frac{d(v)}{q^{2r}}.$$

By the construction of  $W$ , then there exists a nonzero polynomial  $d(t) \in \mathbb{F}_q[t]$  of degree at most  $rN_\phi|S|$  such that

$$(28) \quad w = \phi_{d(t)}(x).$$

Using equations (27) and (28), we obtain

$$\widehat{h}_{U,v}(x) = \frac{\widehat{h}_{U,v}(w)}{\deg(\phi_{d(t)})} > \frac{\frac{d(v)}{q^{2r}}}{q^{r \cdot rN_\phi|S|}},$$

as desired. □

*Proof of Theorem 4.4.* There are two cases.

*Case 1.* The set  $S_x$  is empty.

By Lemma 2.3, all the coefficients  $a_i$  of  $\phi_t$  are from  $F_x$ . Let  $\mathbb{F}_{q^t}$  be a finite field containing all these coefficients.

Assume  $x \in \mathbb{F}_p^{\text{alg}}$ . Let  $\mathbb{F}_{q^{t'v}} = \mathbb{F}_{q^t}(x)$ . Then for every  $n \geq 1$ ,  $\phi_{t^n}(x) \in \mathbb{F}_{q^{t'v}}$ . Because  $\mathbb{F}_{q^{t'v}}$  is finite, there exist distinct positive integers  $n$  and  $n'$  such that  $\phi_{t^n}(x) = \phi_{t^{n'}}(x)$ . Thus  $\phi_{t^{n'-n}}(x) = 0$ ; i.e.  $x \in \phi_{\text{tor}}$ , which is a contradiction with our hypothesis that  $x$  is not a torsion point.

Thus, in *Case 1*,  $x \notin \mathbb{F}_p^{\text{alg}}$ . So,  $x$  is not a constant with respect to the valuations from  $M_{K(x)/F_x}$ . Then, by Theorem 4.15 a), there exists  $v \in M_{K(x)/F_x}$  such that

$$\widehat{h}_v(x) \geq d(v) > q^{-2r}d(v).$$

*Case 2.* The set  $S_x$  is not empty.

Because  $x \notin \phi_{\text{tor}}$ , Theorem 4.15 shows that there exists  $v \in M_{K(x)/F_x}$  such that

$$\widehat{h}_v(x) > q^{-2r-r^2N_\phi|S_x|}d(v) \geq q^{-r(2+(r^2+r)|S_x|)}d(v).$$

□

*Remark 4.21.* Assume that we have a Drinfeld module  $\phi : A \rightarrow K\{\tau\}$  and a non-constant element  $t \in A$  for which  $\phi_t$  is monic. Suppose we are in *Case 1* of the proof of Theorem 4.4. Then that proof shows that for every non-torsion  $x \in K^{\text{alg}}$ , there exists  $v \in M_{K(x)/F_x}$  such that  $\widehat{h}_v(x) \geq \frac{d(v_0)}{[K(x):K]}$ , where  $v_0$  is the place of  $M_{K/F}$  that sits below  $v$ . Because of inequality (9),  $d(v_0) \geq \frac{1}{[K:F(x_1, \dots, x_n)]}$ , where  $\{x_1, \dots, x_n\}$  is the transcendence basis for  $K/F$  with respect to which we constructed the good set of valuations  $M_{K/F}$ . Thus Conjecture 4.1 holds in this case, i.e. when all the coefficients  $a_i$  are from  $\mathbb{F}_p^{\text{alg}}$ , with  $C = \frac{1}{[K:F(x_1, \dots, x_n)]}$ .

With the help of Theorem 4.4 we can get a characterization of the torsion submodule of a Drinfeld module. Let  $K$  be a finitely generated field and let  $\phi : A \rightarrow K\{\tau\}$  be a Drinfeld module. If none of the non-constant  $a \in A$  has the property that  $\phi_a$  is monic, then just pick some non-constant  $t \in A$  and conjugate  $\phi$  by  $\gamma \in K^{\text{alg}} \setminus \{0\}$  such that  $\phi_t^{(\gamma)}$  is monic. Then  $\phi$  and  $\phi^{(\gamma)}$  are isomorphic over  $K(\gamma)$ , which is a finite extension of  $K$  of degree at most  $\deg(\phi_t) - 1$ . So, describing  $\phi_{\text{tor}}(K(\gamma))$  is equivalent with describing  $\phi_{\text{tor}}^{(\gamma)}(K(\gamma))$ . The following result does exactly this. Its proof is immediate after the proof of Theorem 4.15.

**Corollary 4.22.** *Let  $K$  be a finitely generated field and let  $\phi : A \rightarrow K\{\tau\}$  be a Drinfeld module. Let  $t$  be a non-constant element of  $A$ . Let  $\phi_t = \sum_{i=0}^r a_i \tau^i$  and assume that  $a_r = 1$ . Let  $L$  be a finite extension of  $K$  and let  $E$  be the algebraic closure of  $\mathbb{F}_p$  in  $L$ .*

- a) If  $a_0, \dots, a_{r-1} \in E$ , then  $\phi_{\text{tor}}(L) = E$ .*
- b) If not all of the coefficients  $a_0, \dots, a_{r-1}$  are in  $E$ , let  $S = S_L \cap M_{L/E}$ . Let  $b(t) \in \mathbb{F}_q[t]$  be the least common multiple of all the polynomials of degree at most  $rN_\phi|S|$ . Then for all  $x \in \phi_{\text{tor}}(L)$ ,  $\phi_{b(t)}(x) = 0$ .*

*Remark 4.23.* We can also bound the size of the torsion of a Drinfeld module  $\phi$  over a fixed field  $K$  by specializing  $\phi$  at a place of good reduction. This is the classical method used to bound torsion for abelian varieties. The bound that we would obtain by using this more classical method will be much larger than the one from Corollary 4.22 if  $K$  contains a large finite field. However, because our bound is obtained through completely different methods, one can use both methods and then choose the better bound provided by either one.

The bound from Corollary 4.22 *b)* for the torsion subgroup of  $\phi(L)$  is sharp when  $\phi$  is the Carlitz module, as shown by the following example.

**Example 4.24.** For each prime number  $p$  let  $v_\infty : \mathbb{F}_p(t) \setminus \{0\} \rightarrow \mathbb{Z}$  be the valuation such that  $v(b) = -\deg(b)$  for each  $b \in \mathbb{F}_p[t] \setminus \{0\}$ . It is the same notation that we used in Section 2. Also, for each prime number  $p$ , let  $\psi_p$  be the Carlitz module in characteristic  $p$ , i.e.  $\psi_p : \mathbb{F}_p[t] \rightarrow \mathbb{F}_p(t)\{\tau\}$ , given by  $(\psi_p)_t = t\tau^0 + \tau$ .

If  $p = 2$ , we let  $L = \mathbb{F}_2(t)$ . Then with the notation from Corollary 4.22,  $S = \{v_\infty\}$ . Also,  $r = 1$ ,  $N_{\psi_2} = 2$  and so,  $rN_{\psi_2}|S| = 2$ . It is immediate to see that  $\psi_2[t] \subset L$  and also  $\psi_2[1+t] \subset L$ . Thus we do need a polynomial  $b(t)$  of degree 2, i.e.  $b(t) = t^2 + t$ , to kill the torsion of  $\psi_2(L)$ .

If  $p > 2$ , we let  $L = \mathbb{F}_2 \left( (-t)^{\frac{1}{p-1}} \right)$ . Then  $\psi_p[t] \subset L$ . With the notation from Corollary 4.22,  $r = 1$  and  $N_{\psi_p} = 1$ . Also,  $S = \{w_\infty\}$ , where  $w_\infty$  is the unique place of  $L$  sitting above  $v_\infty$ . So, again we see that we need a polynomial  $b(t)$  of degree  $rN_{\psi_p}|S| = 1$  to kill the torsion of  $\psi_p(L)$ .

## 5. THE MORDELL-WEIL THEOREM FOR INFINITELY GENERATED FIELDS

Before stating and proving the theorems from this section we will introduce the notion of modular transcendence degree. This notion refers to the minimal field of definition for a Drinfeld module.

**Definition 5.1.** For a Drinfeld module  $\phi : A \rightarrow K\{\tau\}$ , its field of definition is the smallest subfield of  $K$  containing all the coefficients of  $\phi_a$ , for every  $a \in A$ .

**Lemma 5.2.** *The field of definition of a Drinfeld module is finitely generated.*

*Proof.* Let  $\phi : A \rightarrow K\{\tau\}$ . Let  $t_1, \dots, t_s$  be generators of  $A$  as an  $\mathbb{F}_q$ -algebra. Let  $K_0$  be the field extension of  $\mathbb{F}_q$  generated by all the coefficients of  $\phi_{t_1}, \dots, \phi_{t_s}$ . Then  $K_0$  is finitely generated and by construction,  $K_0$  is the field of definition for  $\phi$ .  $\square$

**Definition 5.3.** Let  $\phi : A \rightarrow K\{\tau\}$  be a Drinfeld module. The modular transcendence degree of  $\phi$  is the minimum transcendence degree over  $\mathbb{F}_p$  of the field of definition for  $\phi^{(\gamma)}$ , where the minimum is taken over all  $\gamma \in K^{\text{alg}} \setminus \{0\}$ .

**Lemma 5.4.** *Let  $\phi : A \rightarrow K\{\tau\}$  be a Drinfeld module and let  $E$  be its field of definition. Let  $t \in A$  be a non-constant element and let  $\phi_t = \sum_{i=0}^r a_i \tau^i$ . Let  $E_0 = \mathbb{F}_p(a_0, \dots, a_r)$  and let  $E_0^{\text{alg}}$  be the algebraic closure of  $E_0$  inside  $K^{\text{alg}}$ . Then  $E_0 \subset E \subset E_0^{\text{alg}}$ .*

*Proof.* Let  $\psi$  be the restriction of  $\phi$  to  $\mathbb{F}_p[t]$ . Clearly,  $\psi$  is defined over  $E_0$ . For every  $a \in A$ ,  $\phi_a$  is an endomorphism of  $\psi$ . Thus for every  $a \in A$ , by Proposition 4.7.4 of [10], the coefficients of  $\phi_a$  are algebraic over  $E_0$ .  $\square$

**Lemma 5.5.** *Let  $\phi : A \rightarrow K\{\tau\}$  be a Drinfeld module. Assume that there exists a non-constant element  $t \in A$  for which  $\phi_t$  is monic. Let  $E$  be the field of definition for  $\phi$ . Then the modular transcendence degree of  $\phi$  is  $\text{trdeg}_{\mathbb{F}_p} E$ .*

*Proof.* By the definition of modular transcendence degree of  $\phi$ , we have to show that for every  $\gamma \in K^{\text{alg}} \setminus \{0\}$ , if  $E^{(\gamma)}$  is the field of definition for  $\phi^{(\gamma)}$ , then

$$(29) \quad \text{trdeg}_{\mathbb{F}_p} E^{(\gamma)} \geq \text{trdeg}_{\mathbb{F}_p} E.$$

Let  $\gamma \in K^{\text{alg}} \setminus \{0\}$ . If  $\phi_t = \sum_{i=0}^r a_i \tau^i$ , then  $\phi_t^{(\gamma)} = \sum_{i=0}^r a_i \gamma^{q^i-1} \tau^i$ .

By Lemma 5.4,  $\text{trdeg}_{\mathbb{F}_p} E = \text{trdeg}_{\mathbb{F}_p} \mathbb{F}_p(a_0, \dots, a_{r-1})$  and

$$\text{trdeg}_{\mathbb{F}_p} E^{(\gamma)} = \text{trdeg}_{\mathbb{F}_p} \mathbb{F}_p(a_0, a_1 \gamma^{q-1}, \dots, a_{r-1} \gamma^{q^{r-1}-1}, \gamma^{q^r-1}).$$

So, in order to prove inequality (29), it suffices to show that

$$(30) \quad \text{trdeg}_{\mathbb{F}_p} \mathbb{F}_p(a_0, a_1 \gamma^{q-1}, \dots, a_{r-1} \gamma^{q^{r-1}-1}, \gamma^{q^r-1}) \geq \text{trdeg}_{\mathbb{F}_p} \mathbb{F}_p(a_0, \dots, a_{r-1}).$$

But,

$$(31) \quad \text{trdeg}_{\mathbb{F}_p} \mathbb{F}_p(a_0, \dots, a_{r-1} \gamma^{q^{r-1}-1}, \gamma^{q^r-1}) = \text{trdeg}_{\mathbb{F}_p} \mathbb{F}_p(a_0, \dots, a_{r-1} \gamma^{q^{r-1}-1}, \gamma).$$

On the other hand,

$$(32) \quad \mathbb{F}_p(a_0, \dots, a_{r-1}) \subset \mathbb{F}_p(a_0, a_1 \gamma^{q-1}, \dots, a_{r-1} \gamma^{q^{r-1}-1}, \gamma).$$

Equations (31) and (32) yield (30). □

**Definition 5.6.** Let  $K_0$  be any subfield of  $K$ . Then the relative modular transcendence degree of  $\phi$  over  $K_0$  is the minimum transcendence degree over  $K_0$  of the compositum field of  $K_0$  and the field of definition of  $\phi^{(\gamma)}$ , the minimum being taken over all  $\gamma \in K^{\text{alg}} \setminus \{0\}$ .

Also, if for some non-constant  $t \in A$ ,  $\phi_t = \sum_{i=0}^r a_i \tau^i$  is monic, we can deduce that the relative modular transcendence degree of  $\phi$  over  $K_0$  can be defined as

$$\text{trdeg}_{K_0} K_0(a_0, \dots, a_{r-1}),$$

as an immediate corollary of Lemma 5.5.

**Theorem 5.7.** *Let  $K$  be a countable field of characteristic  $p$ . Let  $U$  be a coherent good set of valuations on  $K$  and let  $F$  be the field of constants for  $U$ . Let  $\phi : A \rightarrow K\{\tau\}$  be a Drinfeld module of positive relative modular transcendence degree over  $F$ . Then  $\phi(K)$  is a direct sum of a finite torsion submodule and a free submodule of rank  $\aleph_0$ .*

*Proof.* We first recall the definition of a *tame* module. The module  $M$  is tame if every finite rank submodule of  $M$  is finitely generated. According to Proposition 10 from [14], in order to prove Theorem 5.7, it suffices to show that  $\phi(K)$  is a tame module of rank  $\aleph_0$ .

We first prove the following lemma which will allow us to make certain reductions during the proof of Theorem 5.7.

**Lemma 5.8.** *Let  $K'$  be a field extension of  $K$ . Assume that  $\phi(K')$  is a tame module of rank  $\aleph_0$ . Then also  $\phi(K)$  is a tame module of rank  $\aleph_0$ .*

*Proof of Lemma 5.8.* Let  $K_0$  be the field of definition for  $\phi$ . By Lemma 5.2,  $K_0$  is finitely generated. Because  $\phi$  has positive modular transcendence degree,  $\text{trdeg}_{\mathbb{F}_p} K_0 \geq 1$ . Thus, as proved in [18],  $\phi(K_0)$  is a tame module of rank  $\aleph_0$ . Thus  $\phi(K)$  has rank  $\aleph_0$  because both  $\phi(K_0)$  and  $\phi(K')$  have rank  $\aleph_0$ . Because  $\phi(K')$  is tame, then every finite rank submodule of  $\phi(K) \subset \phi(K')$  is finitely generated. Hence  $\phi(K)$  is tame, as desired.  $\square$

Let  $t$  be a non-constant element of  $A$ . Let  $\phi_t = \sum_{i=0}^r a_i \tau^i$ .

Let  $\gamma \in K^{\text{alg}}$  satisfy  $\gamma^{q^r-1} a_r = 1$ . Assume that  $\phi^{(\gamma)}(K(\gamma))$  is a tame module of rank  $\aleph_0$ . Because  $\phi^{(\gamma)}$  is isomorphic to  $\phi$  over  $K(\gamma)$ , it follows that also  $\phi(K(\gamma))$  is a tame module of rank  $\aleph_0$ . Using Lemma 5.8 for  $K' = K(\gamma)$ , we obtain that  $\phi(K)$  is a direct sum of a finite torsion submodule and a free module of rank  $\aleph_0$ . Thus, it suffices to prove Theorem 5.7 under the hypothesis that  $\phi_t$  is monic.

Because  $F$  is the field of constants with respect to  $U$ , then  $F$  is algebraically closed in  $K$ .

Let  $S_0$  be the set of places in  $U$  where  $\phi$  has bad reduction. Because we supposed that  $\phi_t$  is monic, Lemma 4.3 yields that  $S_0$  is the set of places from  $U$  where not all of the coefficients  $a_0, \dots, a_{r-1}$  are integral.

**Lemma 5.9.** *The set  $S_0$  is not empty.*

*Proof of Lemma 5.9.* If  $S_0$  is empty, then by Lemma 2.3,  $a_i \in F$  for all  $i$ . Then by Lemma 5.4, we derive that  $\phi$  is defined over  $F^{\text{alg}} \cap K = F$ , which is a contradiction with our assumption that  $\phi$  has positive relative modular transcendence degree over  $F$ .  $\square$

Because  $S_0$  is not empty, we use Theorem 4.15 b) and conclude that for every non-torsion  $x \in K$ , there exists  $v \in U$  such that

$$(33) \quad \widehat{h}_{U,v}(x) > q^{-r(2+(r^2+r)|S_0|)} d(v).$$

Using inequality (9), we conclude that

$$(34) \quad \widehat{h}_{U,v}(x) > \frac{q^{-r(2+(r^2+r)|S_0|)}}{[K : F(x_1, \dots, x_n)]} =: c(\phi, K) = c > 0.$$

Because  $\widehat{h}_U(x) \geq \widehat{h}_{U,v}(x)$  we conclude that for every non-torsion  $x \in K$ ,

$$(35) \quad \widehat{h}_U(x) > c.$$

On the other hand, Theorem 4.15 b) shows that  $\phi_{\text{tor}}(K)$  is bounded. Moreover, if  $b(t) \in \mathbb{F}_q[t]$  is the least common multiple of all polynomials in  $t$  of degree at most  $(r^2 + r)|S_0|$ , then for every  $x \in \phi_{\text{tor}}(K)$ ,  $\phi_{b(t)}(x) = 0$ .

The last ingredient of our proof is the next lemma.

**Lemma 5.10.** *Let  $R$  be a Dedekind domain and let  $M$  be an  $R$ -module. Assume there exists a function  $h : M \rightarrow \mathbb{R}_{\geq 0}$  satisfying the following properties*

- (i) (triangle inequality)  $h(x \pm y) \leq h(x) + h(y)$ , for every  $x, y \in M$ .
- (ii) if  $x \in M_{\text{tor}}$ , then  $h(x) = 0$ .
- (iii) there exists  $c > 0$  such that for each  $x \notin M_{\text{tor}}$ ,  $h(x) > c$ .
- (iv) there exists  $a \in R \setminus \{0\}$  such that  $R/aR$  is finite and for all  $x \in M$ ,  $h(ax) \geq 4h(x)$ .

*If  $M_{\text{tor}}$  is finite, then  $M$  is a tame  $R$ -module.*

We first show how Lemma 5.10 yields Theorem 5.7 and then prove Lemma 5.10.

Let  $a \in A$  be an element such that  $q^{\deg(\phi_a)} \geq 4$  (we will need this assumption because we will apply next Lemma 5.10). Because of the finiteness of  $\phi_{\text{tor}}(K)$  and because of the equation (35), the Dedekind domain  $A$ ,  $a \in A$ ,  $\phi(K)$  and  $\widehat{h}_U$  satisfy the hypothesis of Lemma 5.10 (note that  $A/aA$  is finite as shown in [14]). We conclude that  $\phi(K)$  is a tame module. Because  $\phi(K)$  is countable, it has at most countable rank. On the other hand, as shown in the proof of Lemma 5.8,  $\phi(K)$  has at least countable rank because  $\phi$  has positive modular transcendence degree. Thus  $\phi(K)$  has rank  $\aleph_0$ . Finally, Proposition 10 of [14] shows that a

tame module of rank  $\aleph_0$  is a direct sum of a finite torsion submodule and a free submodule of rank  $\aleph_0$ .

In order to finish the proof of our Theorem 5.7 we need to prove Lemma 5.10.

*Proof of Lemma 5.10.* By the definition of a tame module, it suffices to assume that  $M$  is a finite rank  $R$ -module and conclude that it is finitely generated.

Let  $a \in R$  as in (iv) of Lemma 5.10. By Lemma 3 of [14],  $M/aM$  is finite (here we use the assumption that  $M_{\text{tor}}$  is finite). The following result is the key to the proof of Lemma 5.10.

*Sublemma 5.11.* For every  $D > 0$ , there exists finitely many  $x \in M$  such that  $h(x) \leq D$ .

*Proof of Sublemma 5.11.* If we suppose Sublemma 5.11 is not true, then we can define

$$C = \inf\{D \mid \text{there exists infinitely many } x \in M \text{ such that } h(x) \leq D\}.$$

Properties (ii) and (iii) and the finiteness of  $M_{\text{tor}}$  yield  $C \geq c > 0$ . By the definition of  $C$ , it must be that there exists an infinite sequence of elements  $z_n$  of  $M$  such that for every  $n$ ,

$$h(z_n) < \frac{3C}{2}.$$

Because  $M/aM$  is finite, there exists a coset of  $aM$  in  $M$  containing infinitely many  $z_n$  from the above sequence.

But if  $k_1 \neq k_2$  and  $z_{k_1}$  and  $z_{k_2}$  are in the same coset of  $aM$  in  $M$ , then let  $y \in M$  be such that  $ay = z_{k_1} - z_{k_2}$ . Using properties (iv) and (i), we get

$$h(y) \leq \frac{h(z_{k_1} - z_{k_2})}{4} \leq \frac{h(z_{k_1}) + h(z_{k_2})}{4} < \frac{3C}{4}.$$

We can do this for any two elements of the sequence that lie in the same coset of  $aM$  in  $M$ . Because there are infinitely many of them lying in the same coset, we can construct infinitely many elements  $z \in M$  such that  $h(z) < \frac{3C}{4}$ , contradicting the minimality of  $C$ .  $\square$

From this point on, our proof of Lemma 5.10 follows the classical descent argument in the Mordell-Weil theorem (see [15]).

Take coset representatives  $y_1, \dots, y_k$  for  $aM$  in  $M$ . Define then

$$B = \max_{i \in \{1, \dots, k\}} h(y_i).$$

Consider the set  $Z = \{x \in M \mid h(x) \leq B\}$ , which is finite according to Sublemma 5.11. Let  $N$  be the finitely generated  $R$ -submodule of  $M$  which is spanned by  $Z$ .

We claim that  $M = N$ . If we suppose this is not the case, then by Sublemma 5.11 we can pick  $y \in M - N$  which minimizes  $h(y)$ . Because  $N$  contains all the coset representatives of  $aM$  in  $M$ , we can find  $i \in \{1, \dots, k\}$  such that  $y - y_i \in aM$ . Let  $x \in M$  be such that  $y - y_i = ax$ . Then  $x \notin N$  because otherwise it would follow that  $y \in N$  (we already know  $y_i \in N$ ). By our choice of  $y$  and by properties (iv) and (i), we have

$$h(y) \leq h(x) \leq \frac{h(y - y_i)}{4} \leq \frac{h(y) + h(y_i)}{4} \leq \frac{h(y) + B}{4}.$$

This means that  $h(y) \leq \frac{B}{3} < B$ . This contradicts the fact that  $y \notin N$  because  $N$  contains all the elements  $z \in M$  such that  $h(z) \leq B$ . This contradiction shows that indeed  $M = N$  and so,  $M$  is finitely generated.  $\square$

Let  $a \in A$  be an element such that  $q^{\deg(\phi_a)} \geq 4$  (we will need this assumption because we will apply next Lemma 5.10). Because of the finiteness of  $\phi_{\text{tor}}(K)$  and because of the equation (35), the Dedekind domain  $A$ ,  $a \in A$ ,  $\phi(K)$  and  $\widehat{h}_U$  satisfy the hypothesis of

Lemma 5.10 (note that  $A/aA$  is finite as shown in [14]). We conclude that  $\phi(K)$  is a tame module. Because  $\phi(K)$  is countable, it has at most countable rank. On the other hand, as shown in the proof of Lemma 5.8,  $\phi(K)$  has at least countable rank because  $\phi$  has positive modular transcendence degree. Thus  $\phi(K)$  has rank  $\aleph_0$ . Proposition 10 of [14] finishes the proof of Theorem 5.7.  $\square$

The following result is an immediate corollary to Theorem 5.7.

**Theorem 5.12.** *Let  $F$  be a countable field of characteristic  $p$  and let  $K$  be a finitely generated field over  $F$ . Let  $\phi : A \rightarrow K\{\tau\}$  be a Drinfeld module of positive relative modular transcendence degree over  $F$ . Then  $\phi(K)$  is a direct sum of a finite torsion submodule and a free submodule of rank  $\aleph_0$ .*

*Proof.* The coherent good set  $U$  of valuations on  $K$  (from the statement of Theorem 5.7) is the set  $M_{K/F}$  constructed in Section 3.  $\square$

The following result gives a structure theorem for Drinfeld modules which are defined over the field of constants (with respect to some coherent good set of valuations).

**Theorem 5.13.** *Let  $F$  be a countable, algebraically closed field of characteristic  $p$  and let  $K$  be a finitely generated extension of  $F$  of positive transcendence degree over  $F$ . If  $\phi : A \rightarrow F\{\tau\}$  is a Drinfeld module, then  $\phi(K)$  is the direct sum of  $\phi(F)$  and a free submodule of rank  $\aleph_0$ .*

*Proof.* Let  $t$  be a non-constant element of  $A$ . Because  $\phi$  is defined over  $F$  and  $F$  is algebraically closed, we can find  $\gamma \in F$  such that  $\phi_t^{(\gamma)}$  is monic. Because  $\phi$  and  $\phi^{(\gamma)}$  are isomorphic

over  $F$ , it suffices to prove Theorem 5.13 for  $\phi^{(\gamma)}$ . Thus we assume from now on that  $\phi_t$  is monic.

We will show next that the module  $\phi(K)/\phi(F)$  is tame.

Let  $\{x_1, \dots, x_n\}$  be a transcendence basis for  $K/F$ . We construct the good set of valuations  $M_{K/F}$  with respect to  $\{x_1, \dots, x_n\}$ , as described in Section 3. Then we construct the local and global heights associated to  $\phi$ .

**Lemma 5.14.** *For every  $x \in F$ ,  $\widehat{h}_{K/F}(x) = 0$ .*

*Proof of Lemma 5.14.* For every  $x \in F$  and for every  $a \in A$ , because  $\phi$  is defined over  $F$ ,  $\phi_a(x) \in F$ . Hence  $v(\phi_a(x)) = 0$  and so, for every  $v \in M_{K/F}$ ,  $\widehat{h}_{K/F,v}(x) = 0$ .  $\square$

We define  $\widehat{H} : \phi(K)/\phi(F) \rightarrow \mathbb{R}_{\geq 0}$  by

$$\widehat{H}(x + \phi(F)) = \widehat{h}_{K/F}(x)$$

for every  $x \in K$ . We will prove in the next lemma that this newly defined function is indeed well-defined.

**Lemma 5.15.** *The function  $\widehat{H}$  is well-defined.*

*Proof of Lemma 5.15.* To show that  $\widehat{H}$  is well-defined, it suffices to show that for every  $x, y \in K$ , if  $x - y = z \in F$ , then  $\widehat{h}_{K/F}(x) = \widehat{h}_{K/F}(y)$ .

Using the triangle inequality and using  $\widehat{h}_{K/F}(z) = 0$  (see Lemma 5.14), we get

$$(36) \quad \widehat{h}_{K/F}(x) \leq \widehat{h}_{K/F}(y) + \widehat{h}_{K/F}(z) = \widehat{h}_{K/F}(y).$$

Similarly, using this time  $\widehat{h}_{K/F}(-z) = 0$  (also  $-z \in F$ ), we get

$$(37) \quad \widehat{h}_{K/F}(y) \leq \widehat{h}_{K/F}(x) + \widehat{h}_{K/F}(-z) = \widehat{h}_{K/F}(x).$$

Inequalities (36) and (37) show that  $\widehat{h}_{K/F}(x) = \widehat{h}_{K/F}(y)$ , as desired.  $\square$

For each  $x \in K$ , we denote by  $\overline{x}$  its image in  $\phi(K)/\phi(F)$ .

**Lemma 5.16.** *The function  $\widehat{H}$  satisfies the properties:*

- (i)  $\widehat{H}(\overline{x+y}) \leq \widehat{H}(\overline{x}) + \widehat{H}(\overline{y})$ , for all  $x, y \in K$ .
- (ii)  $\widehat{H}(\phi_a(\overline{x})) = \deg(\phi_a) \cdot \widehat{H}(\overline{x})$ , for all  $x \in K$  and all  $a \in A \setminus \{0\}$ .
- (iii)  $\widehat{H}(\overline{x}) \geq \frac{1}{[K:F(x_1, \dots, x_n)]}$ , for all  $x \notin F$ .

*Proof of Lemma 5.16.* Properties (i) and (ii) follow immediately from the definition of  $\widehat{H}$  and the fact that  $\phi$  is defined over  $F$  and  $\widehat{h}_{K/F}$  satisfies the triangle inequality and  $\widehat{h}_{K/F}(\phi_a(x)) = \deg(\phi_a) \cdot \widehat{h}_{K/F}(x)$ , for all  $x \in K$  and all  $a \in A \setminus \{0\}$ .

Using the result of Theorem 4.15 part a), we conclude that if  $x \notin F$ , there exists  $v \in M_{K/F}$  such that

$$(38) \quad \widehat{h}_{K/F,v}(x) \geq d(v).$$

Using inequality (9) in (38), we get  $\widehat{h}_{K/F,v}(x) \geq \frac{1}{[K:F(x_1, \dots, x_n)]}$ .

Because  $\widehat{h}_{K/F}(x) \geq \widehat{h}_{K/F,v}(x)$ , we conclude that

$$\widehat{h}_{K/F}(x) \geq \frac{1}{[K : F(x_1, \dots, x_n)]}.$$

$\square$

Now we can finish the proof of Theorem 5.13. The rank of  $\phi(K)/\phi(F)$  is at most  $\aleph_0$  because  $K$  is countable ( $F$  is countable and  $K$  is a finitely generated extension of  $F$ ). We know that  $\phi(K)/\phi(F)$  is torsion-free (if  $\phi_a(x) \in F$  for some  $a \in A \setminus \{0\}$ , then  $x \in F$ , because

$\phi_a \in F\{\tau\}$ ). Because  $\widehat{H}$  satisfies the properties (i)-(iii) from Lemma 5.16, Lemma 5.10 yields that  $\phi(K)/\phi(F)$  is tame.

**Lemma 5.17.** *The rank of  $\phi(K)/\phi(F)$  is  $\aleph_0$ .*

*Proof of Lemma 5.17.* We need to show only that the rank of the above module is at least  $\aleph_0$ . Assume the rank is finite and we will derive a contradiction.

Let  $y_1, \dots, y_g \in K$  be the generators of  $(\phi(K)/\phi(F)) \otimes_A \text{Frac}(A)$  as a  $\text{Frac}(A)$ -vector space. Let  $v \in M_{K/F}$  be a place different from the finitely many places from  $M_{K/F}$  where  $y_1, \dots, y_g$  have poles. Let  $x \in K$  be an element which has a pole at  $v$ . Then for every  $a \in A \setminus \{0\}$ ,  $\phi_a(x)$  has a pole at  $v$ . On the other hand, for every  $a \in A$  and every  $i \in \{1, \dots, g\}$ ,  $\phi_a(y_i)$  is integral at  $v$ . Thus the equation

$$\phi_a(x) = z + \sum_{i=1}^g \phi_{a_i}(y_i)$$

has no solutions  $a, a_1, \dots, a_g \in A$  and  $z \in F$  with  $a \neq 0$ . This provides a contradiction to our assumption that  $y_1, \dots, y_g$  are generators for  $(\phi(K)/\phi(F)) \otimes_A \text{Frac}(A)$  as a  $\text{Frac}(A)$ -vector space.  $\square$

Hence the rank of  $\phi(K)/\phi(F)$  is  $\aleph_0$ . Because  $\phi(K)/\phi(F)$  is tame, Proposition 10 of [14] yields that  $\phi(K)/\phi(F)$  is a direct sum of its torsion submodule and a free submodule of rank  $\aleph_0$ . As explained before,  $\phi(K)/\phi(F)$  is torsion-free. Hence  $\phi(K)/\phi(F)$  is free of rank  $\aleph_0$ . We have the exact sequence:

$$0 \rightarrow \phi(F) \rightarrow \phi(K) \rightarrow \phi(K)/\phi(F) \rightarrow 0.$$

Because  $\phi(K)/\phi(F)$  is free, the above exact sequence splits. Thus,  $\phi(K)$  is a direct sum of  $\phi(F)$  and a free submodule of rank  $\aleph_0$ .  $\square$

The following result is an immediate corollary of Theorem 5.13.

**Theorem 5.18.** *Let  $K$  be a finitely generated field of positive transcendence degree over  $\mathbb{F}_p$ . If  $\phi : A \rightarrow K\{\tau\}$  is a Drinfeld module defined over a finite subfield of  $K$ , then  $\phi(\mathbb{F}_p^{\text{alg}}K)$  is a direct sum of an infinite torsion submodule (which is  $\mathbb{F}_p^{\text{alg}}$ , the entire torsion submodule of  $\phi$ ) and a free submodule of rank  $\aleph_0$ .*

## 6. DRINFELD MODULES OVER THE PERFECT CLOSURE OF A FIELD

In this section we will prove a similar result as Theorem 5.7 valid for the perfect closure of the field  $K$  (as always,  $\phi : A \rightarrow K\{\tau\}$ ). Even though the result is an extension to Theorem 5.7 and the general idea of its proof is similar with the one from Theorem 5.7, it makes more sense to be presented in a separate section. One reason is that it requires more refined height inequalities for Drinfeld modules as the ones proved so far. Also, the results of this section should be seen as an analogue of the author's results from [9] (see also Chapter 3 of [7]). In [9] we proved a Mordell-Weil type theorem for non-isotrivial elliptic curves over the perfect closure of a function field of a curve over a finite field.

The setting for this section is the following:  $K$  is a field of characteristic  $p$  and  $U$  is a coherent good set of valuations on  $K$ . Let  $K_0 \subset K$  be the field of constants with respect to  $U$ .

Let  $\phi : A \rightarrow K\{\tau\}$  be a Drinfeld module. We construct the global height  $\widehat{h}$  and the local heights  $\widehat{h}_v$  with respect to the valuations in  $U$  and the Drinfeld module  $\phi$ .

Assume  $\phi$  has positive relative modular transcendence degree over  $K_0$ . Our goal is to prove there exists a constant  $C > 0$  depending only on  $\phi$  and  $K$  such that for every non-torsion point  $x \in K^{\text{per}}$ ,  $\widehat{h}(x) \geq C$ . Clearly, it suffices to prove our result for an extension  $L$  of  $K$

(as long as we can control the dependence of the constant  $C$  on the field extension). Also, replacing  $\phi$  by an isomorphic Drinfeld module does not affect the validity of our statement. Therefore, we may assume as before, that for some non-constant  $t \in A$ ,  $\phi_t$  is monic.

Let  $\phi_t = \sum_{i=0}^r a_i \tau^i$  (with  $a_r = 1$ ). Let  $S_0$  be the set of places  $v \in U$  for which there exists  $i \in \{0, \dots, r\}$  such that  $v(a_i) < 0$ . By Lemma 4.3,  $S_0$  is the finite set of places  $v \in U$  of bad reduction for  $\phi$ . Not all of the coefficients  $a_i$  are constant, because this would imply  $\phi$  is defined over  $K_0^{\text{alg}} \cap K = K_0$ . Therefore  $S_0$  is not empty.

Let  $L$  be any finite purely inseparable extension of  $K$ . Let  $U_L$  be the set of places of  $L$  which lie above the places from  $U$ . We use the convention, as always, that each valuation function is normalized so that its range equals  $\mathbb{Z}$ . We let  $S$  be the finite set of places  $w \in U_L$  which lie above places  $v \in S_0$ . Then  $|S| = |S_0| > 0$  (above each place from  $S_0$  lies an unique place from  $S$  because  $L/K$  is purely inseparable).

In this section we will use again the definitions of  $M_v$ ,  $P_v$  and  $R_v(\alpha)$  for  $\alpha \in P_v$ . So, we recall that for all  $v \in S$ ,  $|P_v| \leq r + 1$  and for each  $\alpha \in P_v$ ,  $|R_v(\alpha)| \leq q^r$ .

As stated in Lemma 4.14, for every  $x \in L$  and every  $v \in S$ , if  $v(x) \leq 0$ , then either

$$(39) \quad (v(x), \text{ac}_{\pi_v}(x)) \in P_v \times R_v(v(x))$$

$$\text{or } \widehat{\mathbf{h}}_v(x) \geq \frac{-d(v)M_v}{q^r}.$$

Fix  $v \in S$ . Let  $v_0$  be the place of  $S_0$  lying below  $v$ . We define

$$(40) \quad T_v := - \min_{0 \leq i \leq r} \frac{v(a_i)}{q^i}.$$

Because  $v \in S$ ,  $T_v > 0$ . Moreover,  $T_v = -[L : K] \cdot \min_{0 \leq i \leq r} \frac{v_0(a_i)}{q^i}$  (we used that  $L/K$  is purely inseparable and so,  $e(v|v_0) = [L : K]$ ). Thus

$$(41) \quad T_v \geq \frac{[L : K]}{q^r}.$$

Let  $P'_v$  be the set of all  $0 < \alpha \leq T_v$  such that

$$(42) \quad \min_{0 \leq i \leq r} (v(a_i) + q^i \alpha) =: \alpha_1 \in P_v.$$

Because the function  $f(y) = \min_i (v(a_i) + q^i y)$  is piecewise strictly increasing, we conclude  $|P'_v| \leq |P_v| \leq r + 1$  (for each  $\alpha_1 \in P_v$ , there exists at most one  $\alpha \in P'_v$  such that (42) holds).

For each  $\alpha \in P'_v$ , we let  $i_1, \dots, i_l$  be all the indices  $i_j$  such that

$$v(a_{i_j}) + q^{i_j} \alpha = \min_{0 \leq i \leq r} (v(a_i) + q^i \alpha).$$

We let  $R_v(\alpha)$  be the set of all  $\beta$  such that

$$(43) \quad \sum_{1 \leq j \leq l} \text{ac}_{\pi_v}(a_{i_j}) \beta^{q^{i_j}} \in R_v(\alpha_1).$$

Because  $|R_v(\alpha_1)| \leq q^r$ ,  $|R_v(\alpha)| \leq q^{2r}$ .

Let  $P''_v$  be the set of all  $0 < \alpha \leq T_v$  such that  $-\alpha$  is a slope of a segment in the Newton polygon for  $\phi_t$ . For each  $\alpha \in P''_v$ , we let  $i_1, \dots, i_l$  be all the indices  $i_j$  such that  $v(a_{i_j}) + q^{i_j} \alpha = \min_{0 \leq i \leq r} (v(a_i) + q^i \alpha)$ . We let  $R_v(\alpha)$  be the set of all  $\beta$  such that

$$(44) \quad \sum_{1 \leq j \leq l} \text{ac}_{\pi_v}(a_{i_j}) \beta^{q^{i_j}} = 0.$$

We note that it might be that  $\alpha \in P'_v \cap P''_v$ . In that case,  $R_v(\alpha)$  contains all  $\beta$  satisfying both (43) and (44). Therefore  $|R_v(\alpha)| \leq q^{2r} + q^r < q^{2(r+1)}$ .

Let  $Q_v := P_v \cup P'_v \cup P''_v$ . Then  $|Q_v| \leq |P'_v| + |P_v \cup P''_v| \leq (r+1) + (r+1) = 2(r+1)$  (the cardinality of  $P_v \cup P''_v$  is at most  $r+1$  because there are at most  $r$  segments in the Newton polygon for  $\phi_t$  and besides the negatives of the slopes of the segments in the Newton polygon of  $\phi_t$ , only the number 0 might be contained in  $P_v \cup P''_v$ ).

The following result should be seen as an extension of Lemma 4.14.

**Lemma 6.1.** *Let  $v \in S$  and let  $x \in L$ . Assume  $v(x) \leq T_v$ . If  $(v(x), \text{ac}_{\pi_v}(x)) \notin Q_v \times R_v(v(x))$ , then  $\widehat{h}_v(x) \geq \frac{-d(v)M_v}{q^{2r}}$ .*

*Proof.* There are two cases:  $v(x) \leq 0$  and  $0 < v(x) \leq T_v$ .

We analyze the first case:  $v(x) \leq 0$ . Because  $(v(x), \text{ac}_{\pi_v}(x)) \notin P_v \times R_v(v(x))$ , Lemma 4.14 yields  $\widehat{h}_v(x) \geq \frac{-d(v)M_v}{q^r} > \frac{-d(v)M_v}{q^{2r}}$ .

Assume now that  $0 < v(x) \leq T_v$ . Because  $(v(x), \text{ac}_{\pi_v}(x)) \notin P'_v \times R_v(v(x))$ ,  $v(\phi_t(x)) = \min_{0 \leq i \leq r} v(a_i x^{q^i})$  (see the remark at the end of the proof of Lemma 4.8). Because  $v(x) \leq T_v$ ,  $v(a_i x^{q^i}) \leq 0$ , for some  $0 \leq i \leq r$  (see the definition of  $T_v$  from (40)). Hence  $v(\phi_t(x)) \leq 0$ . Let  $i_1, \dots, i_l \in \{0, \dots, r\}$  be all the indices  $i_j$  such that  $v(\phi_t(x)) = v(a_{i_j} x^{q^{i_j}})$ . Then  $\text{ac}_{\pi_v}(\phi_t(x)) = \sum_j \text{ac}_{\pi_v}(a_{i_j}) \text{ac}_{\pi_v}(x)^{q^{i_j}}$ . Because  $(v(x), \text{ac}_{\pi_v}(x)) \notin P'_v \times R_v(v(x))$ , we conclude

$$(45) \quad (v(\phi_t(x)), \text{ac}_{\pi_v}(\phi_t(x))) \notin P_v \times R_v(v(\phi_t(x))).$$

Because  $v(\phi_t(x)) \leq 0$ , Lemma 4.14 yields  $\widehat{h}_v(\phi_t(x)) \geq \frac{-d(v)M_v}{q^r}$ . Hence  $\widehat{h}_v(x) = \frac{\widehat{h}_v(\phi_t(x))}{q^r} \geq \frac{-d(v)M_v}{q^{2r}}$ , as desired.  $\square$

The proof of following result is similar with the proof of Theorem 4.15.

**Lemma 6.2.** *Let  $x \in L$ . Then either there exists  $v \in S$  such that  $\widehat{h}_v(x) \geq \frac{-d(v)M_v}{q^{4r(r+1)^2|S|+2r}}$ , or there exists a polynomial  $b \in \mathbb{F}_q[t]$  of degree at most  $4(r+1)^2|S|$  such that for every  $v \in S$ ,  $v(\phi_b(x)) > T_v$ .*

*Proof.* Let  $v \in S$ . We apply Lemma 4.16 with  $N = T_v$ ,  $I = Q_v$  and  $R(\alpha) = R_v(\alpha)$  for every  $\alpha \in Q_v$ . Because  $|Q_v| \leq 2(r+1)$  and  $|R_v(\alpha)| < q^{2(r+1)}$ , for every  $\alpha \in Q_v$ , we conclude that the following is true.

**Fact 6.3.** *Let  $v \in S$ . Let  $W$  be an  $\mathbb{F}_q$ -vector subspace of  $L$  with the property that for all  $w \in W$ ,  $(v(w), \text{ac}_{\pi_v}(w)) \in Q_v \times R_v(v(w))$  whenever  $v(w) \leq T_v$ .*

*Then the  $\mathbb{F}_q$ -codimension of  $\{w \in W \mid v(w) > T_v\}$  in  $W$  is bounded above by  $4(r+1)^2$ .*

We apply Fact 6.3 for each  $v \in S$  and we deduce the following two results.

**Fact 6.4.** *Let  $W$  be an  $\mathbb{F}_q$ -subspace of  $L$  such that  $(v(w), \text{ac}_{\pi_v}(w)) \in Q_v \times R_v(v(w))$  whenever  $v \in S$ ,  $w \in W$  and  $v(w) \leq T_v$ . Then the  $\mathbb{F}_q$ -codimension of*

$$\{w \in W \mid v(w) > T_v \text{ for all } v \in S\}$$

*in  $W$  is bounded above by  $4(r+1)^2|S|$ .*

**Fact 6.5.** *Let notation be as in Fact 6.4. If moreover,  $\dim_{\mathbb{F}_q} W > 4(r+1)^2|S|$ , then there exists a nonzero  $w \in W$  such that  $v(w) > T_v$  for all  $v \in S$ .*

We are now ready to finish the proof of Lemma 6.2.

Let  $W = \text{Span}_{\mathbb{F}_q}(\{x, \phi_t(x), \dots, \phi_{t^{4(r+1)^2|S|}}(x)\})$ . Because  $\dim_{\mathbb{F}_q} W = 4(r+1)^2|S| + 1$ , Fact 6.5 yields the existence of a nonzero  $w \in W$  such that either there exists  $v \in S$  such that

$$(46) \quad v(w) \leq T_v \text{ and } (v(w), \text{ac}_{\pi_v}(w)) \notin Q_v \times R_v(v(w)),$$

or for all  $v \in S$ ,

$$(47) \quad v(w) > T_v.$$

If (46) holds, then by Lemma 6.1,  $\widehat{h}_v(w) \geq \frac{-d(v)M_v}{q^{2r}}$ . Because  $w \in W \setminus \{0\}$ , there exists a nonzero polynomial  $b \in \mathbb{F}_q[t]$  of degree at most  $4(r+1)^2|S|$  such that  $w = \phi_b(x)$ . Thus

$$\widehat{h}_v(x) = \frac{\widehat{h}_v(w)}{\deg(\phi_b)} \geq \frac{-d(v)M_v}{q^{4r(r+1)^2|S|+2r}}.$$

If (47) holds, then  $v(\phi_b(x)) > T_v$  for every  $v \in S$  with  $\deg(b) \leq 4(r+1)^2|S|$ , where  $w = \phi_b(x)$ . □

The next theorem is the main step in order to prove the existence of a positive lower bound  $C$  for the height of non-torsion points in  $K^{\text{per}}$  for a Drinfeld modules  $\phi : A \rightarrow K\{\tau\}$  of positive relative modular transcendence degree.

**Theorem 6.6.** *Let  $K$  be a field of characteristic  $p$  and let  $U$  be a coherent good set of valuations on  $K$ . Let  $K_0 \subset K$  be the field of constants with respect to the valuations in  $U$ .*

*Let  $\phi : A \rightarrow K\{\tau\}$  be a Drinfeld module of positive relative modular transcendence degree over  $K_0$ . Let  $t \in A$  be a non-constant element and assume  $\phi_t = \sum_{i=0}^r a_i \tau^i$  is monic. Let  $S_0$  be the set of the places in  $U$  of bad reduction for  $\phi$ . Let  $s = |S_0|$ .*

*Then for every non-torsion point  $x \in K^{\text{per}}$ ,  $\widehat{h}(x) > \frac{\min_{v_0 \in S_0} d(v_0)}{q^{4r(r+1)^2s+3r}} > 0$ .*

Before proving Theorem 6.6 we show how this theorem implies a more general result.

**Theorem 6.7.** *Let  $K$  be a field of characteristic  $p$  and let  $U$  be a coherent good set of valuations on  $K$ . Let  $K_0 \subset K$  be the field of constants with respect to the valuations in  $U$ .*

*Let  $\phi : A \rightarrow K\{\tau\}$  be a Drinfeld module of positive relative modular transcendence degree*

over  $K_0$ . There exists a positive constant  $C$  depending only on  $\phi$  and  $K$  such that for every non-torsion  $x \in K^{\text{per}}$ ,  $\widehat{h}(x) > C$ .

*Proof.* Let  $t \in A$  be a non-constant element minimizing  $\deg(\phi_a)$  as  $a$  ranges over non-constant elements of  $A$ . Let  $\gamma \in K^{\text{alg}}$  such that  $\phi_t^{(\gamma)}$  is monic. Let  $L = K(\gamma)$ . If  $\deg(\phi_t) = q^r$ , then  $[L : K] \leq q^r - 1$ . As explained before,  $\phi$  and  $\phi^{(\gamma)}$  are isomorphic and  $\widehat{h}_\phi = \widehat{h}_{\phi^{(\gamma)}}$ .

Let  $S$  be the set of all places of  $L$  lying above places in  $S_0$  (we use the notation from Theorem 6.6). Then

$$(48) \quad |S| \leq (q^r - 1)|S_0|,$$

because  $[L : K] \leq q^r - 1$ . Also, for each  $v \in S$ ,

$$(49) \quad d(v) = \frac{d(v_0)f(v|v_0)}{[L : K]} \geq \frac{d(v_0)}{[L : K]}.$$

By Theorem 6.6 applied to  $\phi^{(\gamma)} : A \rightarrow L\{\tau\}$ , for every  $x \in K^{\text{per}} \subset L^{\text{per}}$ ,

$$(50) \quad \widehat{h}_\phi(x) = \widehat{h}_{\phi^{(\gamma)}}(x) \geq \frac{\min_{v \in S} d(v)}{q^{4r(r+1)^2|S|+3r}} =: C > 0.$$

Using (48) and (49) in (50), we see that the positive constant  $C$  is bounded below by another positive constant which depends only on  $\phi$  and  $K$ , as desired.  $\square$

*Proof of Theorem 6.6.* We first recall that because  $\phi$  has positive relative modular transcendence degree over  $K_0$ ,  $s \geq 1$ .

Let  $x \in K^{\text{per}}$  be a non-torsion point and let  $L = K(x)$ . Let  $S$  be the set of places of  $L$  which lie above places in  $S_0$ . Because  $L \subset K^{\text{per}}$ ,  $|S| = s$ .

Let  $U_L$  be the set of places of  $L$  which lie above the places from  $U$  (as always, they are normalized so that the range of each valuation function is  $\mathbb{Z}$ ). For each place  $v \in U_L$ , we

denote by  $v_0 \in S_0$  the corresponding place which lies below  $v$ . Because  $L/K$  is a purely inseparable extension, for each place  $v \in U_L$ ,

$$(51) \quad d(v)e(v|v_0) = \frac{f(v|v_0)d(v_0)e(v|v_0)}{[L : K]} = \frac{d(v_0)[L : K]}{[L : K]} = d(v_0).$$

On the other hand, by its definition,  $M_v = e(v|v_0) \min_{0 \leq i < r} \frac{v_0(a_i)}{q^r - q^i}$  and so, if  $v \in S$ ,

$$(52) \quad M_v < \frac{-e(v|v_0)}{q^r}.$$

Using (51) in (52) we get  $-d(v)M_v > \frac{d(v_0)}{q^r}$ .

If there exists  $v \in S$  such that  $\widehat{h}_v(x) > \frac{d(v_0)}{q^{4r(r+1)^2s+3r}}$ , then

$$\widehat{h}(x) \geq \widehat{h}_v(x) > \min_{v_0 \in S_0} \frac{d(v_0)}{q^{4r(r+1)^2s+3r}},$$

as desired. Therefore, assume from now on, that for each  $v \in S$ ,

$$\widehat{h}_v(x) \leq \frac{d(v_0)}{q^{4r(r+1)^2s+3r}} < \frac{-d(v)M_v}{q^{4r(r+1)^2s+2r}}.$$

Then by Lemma 6.2, there exists a nonzero polynomial  $b \in \mathbb{F}_q[t]$  of degree at most  $4(r+1)^2s$  such that

$$(53) \quad v(\phi_b(x)) > T_v$$

for all  $v \in S$ . Because  $b \neq 0$  and  $x \notin \phi_{\text{tor}}$ ,  $y := \phi_b(x) \neq 0$ . Because  $U$  is a coherent good set of valuations,  $U_L$  is a good set of valuations on  $L$  and so, because  $y \neq 0$ ,

$$(54) \quad \sum_{v \in U_L} d(v) \cdot v(y) = 0.$$

By its definition, for each  $v \in S$ ,  $T_v = -e(v|v_0) \min_{0 \leq i \leq r} \frac{v_0(a_i)}{q^i} \geq \frac{e(v|v_0)}{q^r}$ . Hence, using (53) and (51), we get

$$(55) \quad \sum_{v \in S} d(v) \cdot v(y) > \sum_{v \in S} \frac{d(v)e(v|v_0)}{q^r} = \sum_{v_0 \in S_0} \frac{d(v_0)}{q^r}.$$

Using (55) in (54), we conclude there exists a finite set  $U(y)$  of places in  $U_L \setminus S$  such that for each  $v \in U(y)$ ,  $v(y) < 0$  and moreover

$$(56) \quad \sum_{v \in U(y)} d(v) \cdot v(y) < - \sum_{v_0 \in S_0} \frac{d(v_0)}{q^r}.$$

For each  $v \in U(y)$ , because  $v \notin S$  and  $v(y) < 0$ , Lemma 4.10 yields

$$\widehat{h}_v(y) = -d(v) \cdot v(y).$$

Using (56) we conclude

$$(57) \quad \sum_{v \in U(y)} \widehat{h}_v(y) > \sum_{v_0 \in S_0} \frac{d(v_0)}{q^r}.$$

Inequality (57) yields

$$(58) \quad \widehat{h}(y) > \sum_{v_0 \in S_0} \frac{d(v_0)}{q^r}.$$

Because  $y = \phi_b(x)$  and the degree of  $b$  as a polynomial in  $t$  is at most  $4(r+1)^2s$ , we get

$$\widehat{h}(x) \geq \frac{\widehat{h}(y)}{q^{4r(r+1)^2s}} > \sum_{v_0 \in S_0} \frac{d(v_0)}{q^{4r(r+1)^2s+r}} > \min_{v_0 \in S_0} \frac{d(v_0)}{q^{4r(r+1)^2s+3r}},$$

as desired. □

*Remark 6.8.* Theorem 6.6 is sharp in the sense that if we assume  $\phi : A \rightarrow K_0\{\tau\}$  and we keep the rest of the assumptions from Theorem 6.6, then the conclusion of Theorem 6.6 fails. Indeed, let  $x \in K \setminus K_0$ . By Lemma 4.10,  $\widehat{h}(x) = - \sum_{v \in U} d(v) \cdot \min\{0, v(x)\}$ . Because

$x \notin K_0$ ,  $\widehat{h}(x) > 0$ . Moreover, for each  $n \geq 1$ ,  $x^{1/p^n} \in K^{\text{per}}$  and an easy computation, using again Lemma 4.10 shows  $\widehat{h}(x^{1/p^n}) = \frac{\widehat{h}(x)}{p^n}$ . Therefore, as  $n$  goes to infinity, the height of  $x^{1/p^n}$  goes strictly decreasing to 0. Hence there is no uniform positive lower bound for the height of a non-torsion point in  $K^{\text{per}}$ .

**Corollary 6.9.** *Let  $K$  be a finitely generated field of characteristic  $p$ . Let  $\phi : A \rightarrow K\{\tau\}$  be a Drinfeld module of positive modular transcendence degree. There exists a constant  $C > 0$  depending only on  $\phi$  and  $K$  such that for every non-torsion point  $x \in K^{\text{per}}$ ,  $\widehat{h}(x) \geq C$ .*

*Proof.* Let  $V$  be a projective normal variety defined over a finite field, whose function field is  $K$ . We construct the coherent good set  $U$  of valuations on  $K$  associated to the irreducible divisors of  $V$  (in Section 3 we presented a completely algebraic construction of  $U$ ). The field of constants with respect to  $U$  is the maximal finite subfield of  $K$ . Because  $\phi$  has positive modular transcendence degree we can apply Theorem 6.7 and get the existence of the constant  $C$  in Corollary 6.9.  $\square$

Using Theorem 6.6 we prove the following Mordell-Weil type theorem.

**Theorem 6.10.** *Let  $K$  be a countable field of characteristic  $p$  and let  $U$  be a coherent good set of valuations on  $K$ . Let  $K_0 \subset K$  be the field of constants with respect to the valuations from  $U$ . Let  $\phi : A \rightarrow K\{\tau\}$  be a Drinfeld module of positive relative modular transcendence degree over  $K_0$ . Then  $\phi(K^{\text{per}})$  is the direct sum of a finite torsion submodule with a free submodule of rank  $\aleph_0$ .*

*Proof.* We will prove  $\phi(K^{\text{per}})$  has rank  $\aleph_0$  and is a tame module. According to Proposition 10 of [14], these two properties yield our conclusion.

As proved in Lemma 5.8, it suffices to prove our theorem after replacing  $K$  by a finite extension. Therefore, we assume from now on that there exists a non-constant  $t \in A$  such that  $\phi_t$  is monic. Let  $q^r$  be the degree of  $\phi_t$ .

We know that  $\phi(K^{\text{per}})$  has rank at most  $\aleph_0$  because  $K^{\text{per}}$  is countable, as  $K$  is countable. On the other hand,  $\phi(K^{\text{per}})$  has at least rank  $\aleph_0$  because  $\phi$  has positive modular transcendence degree (see the proof of Lemma 5.8).

In order to show  $\phi(K^{\text{per}})$  is tame, we use Lemma 5.10. Thus we need to show that  $\phi_{\text{tor}}(K^{\text{per}})$  is finite. The other conditions of the above mentioned lemma are already satisfied by the global height function associated to  $\phi$  (see Theorem 6.6) and by any element  $a \in A$  of degree at least 2 (so that  $\deg(\phi_a) = q^{\deg(a)} \geq 4$ ). Because  $K^{\text{per}} = \bigcup_{n \geq 1} K^{1/p^n}$ , it suffices to show  $\phi_{\text{tor}}(K^{1/p^n})$  is uniformly bounded.

Let  $s \geq 1$  be the number of places in  $U$  of bad reduction for  $\phi$ . Let  $U_n$  be the good set of places on  $K^{1/p^n}$ , which lie above places in  $U$ . There exists exactly one place in  $U_n$  lying above each place in  $U$  because  $K^{1/p^n}/K$  is a purely inseparable extension. Thus, for each  $n \geq 1$ , there are  $s$  places of bad reduction for  $\phi$  in  $U_n$ . By Theorem 4.15, the size of  $\phi_{\text{tor}}(K^{1/p^n})$  is bounded above in terms of  $q$ ,  $r$  and  $s$ , independently of  $n$ . Hence  $\phi_{\text{tor}}(K^{\text{per}})$  is finite. As explained in the previous paragraph, Lemma 5.10 concludes the proof of our theorem.  $\square$

Just as Corollary 6.9 followed from Theorem 6.6, in the same way we can deduce the following result from Theorem 6.10.

**Corollary 6.11.** *Let  $K$  be a finitely generated field of characteristic  $p$  and let  $\phi : A \rightarrow K\{\tau\}$  be a Drinfeld module of positive modular transcendence degree. Then  $\phi(K^{\text{per}})$  is a direct sum of a finite torsion submodule with a free submodule of rank  $\aleph_0$ .*

*Remark 6.12.* Corollary 6.11 is sharp in the sense that we cannot drop the hypothesis that  $\phi$  has positive modular transcendence degree. For example, let  $\phi : \mathbb{F}_q[t] \rightarrow \mathbb{F}_q(t)\{\tau\}$  be given by  $\phi_t = \tau$ . Then we can check immediately that  $\phi(\mathbb{F}_q(t)^{\text{per}})$  is a direct sum of a finite torsion submodule with a free  $\mathbb{F}_q[t, t^{-1}]$ -submodule of rank  $\aleph_0$ .

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